

Quadratic BSDEs with jumps and related non-linear expectations: a fixed-point approach*

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October 29, 2012

Abstract

We prove the existence of bounded solutions of quadratic backward SDEs with jumps, using a direct fixed point approach as in Tevzadze [36]. Under an additional standard assumption, we prove a uniqueness result, thanks to a comparison theorem. Then we study the properties of the corresponding g -expectations, we obtain in particular a non linear Doob-Meyer decomposition for g -submartingales and their regularity in time. As a consequence of this results, we obtain a converse comparison theorem for our class of BSDEs. We give applications for dynamic risk measures and their dual representation, and compute their inf-convolution, with some explicit examples.

Key words: BSDEs, quadratic growth, jumps, non-linear Doob-Meyer decomposition, dynamic risk measures, inf-convolution.

AMS 2000 subject classifications: 60H10, 60H30

*Research supported by the Chair *Financial Risks* of the *Risk Foundation* sponsored by Société Générale, the Chair *Derivatives of the Future* sponsored by the Fédération Bancaire Française, and the Chair *Finance and Sustainable Development* sponsored by EDF and Calyon.

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1 Introduction

Motivated by duality methods and maximum principles for optimal stochastic control, Bismut studied in [6] a linear backward stochastic differential equation (BSDE). In their seminal paper [30], Pardoux and Peng generalized such equations to the non-linear Lipschitz case and proved existence and uniqueness results in a Brownian framework. Since then, a lot of attention has been given to BSDEs and their applications, not only in stochastic control, but also in theoretical economics, stochastic differential games and financial mathematics.

In this context, the generalization of Backward SDEs to a setting with jumps enlarges again the scope of their applications, for instance to insurance modeling, in which jumps are inherent. Li and Tang [24] first proved the wellposedness of Lipschitz BSDEs with jumps, using a fixed point approach similar to the one used in [30].

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ generated by an \mathbb{R}^d -valued Brownian motion B and a random measure μ with compensator ν , solving a BSDE with generator g , and terminal condition ξ consists in finding a triple of progressively measurable processes (Y, Z, U) such that for all $t \in [0, T]$, $\mathbb{P} - a.s.$

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_{\mathbb{R}^d \setminus \{0\}} U_s(x) (\mu - \nu)(ds, dx). \quad (1.1)$$

See Section 2.1 for more precise definitions and notations.

In this paper, g will be supposed to satisfy a Lipschitz-quadratic growth property. More precisely, g will be Lipschitz in y , and will satisfy the quadratic growth condition (iii) of Assumption 2.1.

When the filtration is generated only by a Brownian motion, the existence and uniqueness of quadratic BSDEs with a bounded terminal condition has been first treated in Kobylanski [19], by means of an exponential transform method, allowing to fall into the scope of BSDEs with a coefficient having a linear growth, then the result for quadratic BSDEs is obtained by an approximation operation. Tevzadze [36] has also given a direct proof for the wellposedness in the Lipschitz-quadratic setting. His methodology is fundamentally different, since he uses fixed-point approach to obtain existence of a solution for small terminal condition, and then pastes solutions together in the general bounded case. More recently, Briand and Hu [7] went beyond the bounded case to obtain an existence result for quadratic BSDEs with a terminal condition having finite exponential moments. In a similar vein, but using a forward point of view and stability results for special class of quadratic semimartingales, Barrieu and El Karoui [4] generalized the above results.

Nonetheless, when it comes to quadratic BSDEs in a discontinuous setting, the literature is less abounding. Until very recently, the only existing results concerned particular cases of quadratic BSDEs, mainly with links to applications to utility maximization or indifference pricing problems. Thus, Becherer [5] studied first bounded solutions to BSDEs with jumps in a finite activity setting, and his general results were improved by Morlais [28], who proved existence of the solution to a special quadratic BSDE with jumps, using the same type of techniques as Kobylanski. More recently, Ngupeyou [29], in his PhD thesis, extended partly the approach of [4] to the jump case. After the completion of this paper, we became aware of a very recent result of Laeven and Stadje [20] who proved a general existence result for BSDEJs with convex generators, using verification arguments. We emphasize that our approach is very different and do not need any convexity assumption, even though the two results do not imply each other.

Our first aim in this paper is to extend the fixed-point methodology of Tevzadze [36] to the case of discontinuous filtration. We first obtain our result for a terminal condition ξ having a $\|\cdot\|_\infty$ -norm which is small enough. Then the result for any ξ in \mathbb{L}^∞ follows by splitting ξ in pieces having a small enough norm, and then pasting the obtained solutions to a single equation. Since we deal with bounded solutions, the space of BMO martingales will play a particular role in our setting, this will be the natural space for the continuous and the pure jump martingale terms appearing in the BSDE 1.1, when Y is bounded.

In this framework with jumps, we need additional assumptions on the generator g for a comparison theorem to hold. Namely, we will use the Assumption 2.5, first introduced by Royer [34] in order to ensure the validity of a comparison theorem for Lipschitz BSDEs with jumps. We extend here this comparison theorem to our setting (Proposition 2.7), and then use it to give a uniqueness result.

This wellposedness result for bounded quadratic BSDEs with jumps opens the way to many possible applications. Barrieu and El Karoui [3] used quadratic BSDEs to define time consistent convex risk measures and study their properties. We extend here some of these results to the case with jumps. When the generator g is independent of y and convex in (z, u) , we can define through the solution of the BSDE a convex operator acting on the terminal condition. This operator, called g -expectation, has been first studied by Peng [31], and then extended to the case of quadratic coefficients by Ma and Yao [26], or to discontinuous filtrations by Royer [34] and Lin [25].

In this paper, we go further in the study of quadratic BSDEs with jumps by proving a non-linear Doob Meyer decomposition for g -submartingales. As a consequence, we obtain a converse comparison theorem.

The last two results are dedicated to an explicit dual representation of the solution Y , when g is independent of y and convex in (z, u) . This allows to study some particular risk measures on a discontinuous filtration, like the *entropic risk measure*, corresponding to the solution of a quadratic BSDE. Finally, we prove an explicit representation for the inf-convolution of quadratic BSDEs, thus giving the form of the optimal risk transfer between two agents using quadratic convex g -expectations as risk measures. The inf-convolution is again a convex operator, solving a particular BSDE. We give a sufficient condition for this BSDE to have a coefficient satisfying a quadratic growth property.

The rest of this paper is organized as follows. In Section 2, we obtain our general existence and uniqueness result, then in Section 3, we study general properties of quadratic g -martingales with jumps, such as regularity in time and the Doob-Meyer decomposition. Finally, Section 4 is devoted to the two applications mentioned above.

2 An existence and uniqueness result

2.1 Notations

Given a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$ generated by a \mathbb{R}^d -valued Brownian motion B , solving a BSDE with generator g , and terminal condition ξ consists in finding a pair of progressively measurable processes (Y, Z) such that

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad \mathbb{P} - a.s., \quad t \in [0, T]. \quad (2.1)$$

The process Y we define this way is a possible generalization of the conditional expectation of ξ ,

since when g is the null function, we have $Y_t = \mathbb{E}^\mathbb{P}[\xi|\mathcal{F}_t]$, and in this case, Z is the process appearing in the (\mathcal{F}_t) -martingale representation property of $\{\mathbb{E}^\mathbb{P}[\xi|\mathcal{F}_t], t \geq 0\}$.

In the case of a filtered probability space generated by both a Brownian motion B and a Poisson random measure μ with compensator ν , the martingale representation for $\{\mathbb{E}^\mathbb{P}[\xi|\mathcal{F}_t], t \geq 0\}$ becomes

$$\mathbb{E}^\mathbb{P}[\xi|\mathcal{F}_t] = \int_0^t Z_s dB_s + \int_0^t \int_{\mathbb{R}^d \setminus \{0\}} U_s(x) \tilde{\mu}(dx, ds),$$

where U is a predictable function, and $\tilde{\mu} = \mu - \nu$.

This leads to the following natural generalization of equation (2.1) to the case with jumps.

Definition 2.1. *Let ξ be a \mathcal{F}_T -measurable random variable. A solution of the BSDEJ with terminal condition ξ and generator g is a triple (Y, Z, U) of progressively measurable processes such that*

$$Y_t = \xi + \int_t^T g_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_{\mathbb{R}^d \setminus \{0\}} U_s(x) \tilde{\mu}(dx, ds), \quad 0 \leq t \leq T, \quad \mathbb{P} - a.s. \quad (2.2)$$

where $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{A}(E) \rightarrow \mathbb{R}$ is a given application and

$$\mathcal{A}(E) := \left\{ u : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}, \mathcal{B}(\mathbb{R}^d \setminus \{0\}) - \text{measurable} \right\}.$$

The processes Z and U are supposed to satisfy the minimal assumptions so that the quantities in (2.2) are well defined, namely $(Z, U) \in \mathcal{Z} \times \mathcal{U}$, where \mathcal{Z} denotes the space of all \mathbb{F} -predictable \mathbb{R}^d -valued processes Z with

$$\int_0^T |Z_t|^2 dt < +\infty, \quad \mathbb{P} - a.s.$$

and \mathcal{U} denotes the space of all \mathbb{F} -predictable functions U with

$$\int_0^T \int_{\mathbb{R}^d \setminus \{0\}} |U_s(x)|^2 \nu_t(dx) ds < +\infty, \quad \mathbb{P} - a.s.$$

Notice that in this discontinuous setting, the generator g depends on both Z and U . Here U plays a role analogous to the quadratic variation in the continuous case. However, there are some notable differences, since for each t , U_t is a function from E into $\mathbb{R}^d \setminus \{0\}$, and that is why the treatment of the dependence in u in the assumptions for the generator is not symmetric to the treatment of the dependence in z , and in particular we deal with Fréchet derivatives with respect to u . See for instance Assumption 2.3.

When we pass from the continuous setting to the discontinuous one, the natural analogue of the integral with respect to the Brownian motion would be the integral with respect to a compensated Poisson random measure. The compensator would not depend on t , nor on ω , by comparison to the non dependence in ω of the quadratic variation of the Brownian motion.

However we will allow the compensator of the jump measure that we will consider to depend on both t and ω . This will not increase the complexity of our proofs, provided that the martingale representation property of Assumption 2.2 holds true.

We consider in all the paper a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, whose filtration satisfies the usual hypotheses of completeness and right-continuity. We suppose that this filtration is generated by a d -dimensional Brownian motion B and an independent integer valued random measure $\mu(\omega, dt, dx)$ defined on $\mathbb{R}^+ \times E$, with compensator $\lambda(\omega, dt, dx)$, where $E := \mathbb{R}^d \setminus \{0\}$. $\tilde{\Omega} :=$

$\Omega \times \mathbb{R}^+ \times E$ is equipped with the sigma-field $\tilde{\mathcal{P}} := \mathcal{P} \times \mathcal{E}$, where \mathcal{P} denotes the predictable σ -field on $\Omega \times \mathbb{R}^+$ and \mathcal{E} is the Borel σ -field on E .

To guarantee the existence of the compensator $\lambda(\omega, dt, dx)$, we assume that for each A in $\mathcal{B}(E)$ and each ω in Ω , the process $X_t := \mu(\omega, A, [0, t]) \in \mathcal{A}_{loc}^+$, which means that there exists an increasing sequence of stopping times (T_n) such that $T_n \rightarrow +\infty$ a.s. and the stopped processes $X_t^{T_n}$ are increasing, càdlàg, adapted and satisfy $\mathbb{E}[X_\infty] < +\infty$.

We assume in all the paper that λ is absolutely continuous with respect to the Lebesgue measure dt , i.e. $\lambda(\omega, dt, dx) = \nu_t(\omega, dx)dt$. Finally, we denote $\tilde{\mu}$ the compensated jump measure:

$$\tilde{\mu}(\omega, dx, dt) = \mu(\omega, dx, dt) - \nu_t(\omega, dx) dt.$$

2.2 Standard spaces and norms

We introduce the following norms and spaces for any $p \geq 1$.

\mathcal{S}^p is the space of \mathbb{R} -valued càdlàg and \mathcal{F}_t -progressively measurable processes Y such that

$$\|Y\|_{\mathcal{S}^p}^p := \mathbb{E}^{\mathbb{P}} \left[\sup_{0 \leq t \leq T} Y_t^p \right] < +\infty.$$

\mathcal{S}^∞ is the space of \mathbb{R} -valued càdlàg and \mathcal{F}_t -progressively measurable processes Y such that

$$\|Y\|_{\mathcal{S}^\infty} := \sup_{0 \leq t \leq T} \|Y_t\|_\infty < +\infty.$$

\mathbb{H}^p is the space of \mathbb{R}^d -valued and \mathcal{F}_t -progressively measurable processes Z such that

$$\|Z\|_{\mathbb{H}^p}^p := \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Z_t|^2 dt \right)^{\frac{p}{2}} \right] < +\infty.$$

The three spaces above are the classical ones in the BSDE theory in continuous filtrations. We introduce finally a space which is specific to the jump case, and which plays the same role for U as \mathbb{H}^p for Z .

\mathbb{J}^p is the space of predictable and \mathcal{E} -measurable applications $U : \Omega \times [0, T] \times E$ such that

$$\|U\|_{\mathbb{J}^p}^p := \mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T \int_E |U_s(x)|^2 \nu_s(dx) ds \right)^{\frac{p}{2}} \right] < +\infty.$$

2.3 A word on càdlàg BMO martingales

The recent literature on quadratic BSDEs is very rich on remarks and comments about the very deep theory of continuous BMO martingales. However, it is clearly not as well documented when it comes to càdlàg BMO martingales, whose properties are crucial ingredients in this paper. Indeed, apart from some remarks in the book by Kazamaki [17], the extension to the càdlàg case of the classical results of BMO theory, cannot always be easily find. Our main goal in this short subsection is to give a rapid overview of the existing literature and results concerning BMO martingales with càdlàg trajectories, with an emphasis where the results differ from the continuous case. Let us start by recalling some notations and definitions.

BMO is the space of square integrable càdlàg \mathbb{R}^d -valued martingales M such that

$$\|M\|_{\text{BMO}} := \sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}_\tau^\mathbb{P} \left[(M_T - M_{\tau-})^2 \right] \right\|_\infty < +\infty,$$

where \mathcal{T}_0^T is the set of \mathcal{F}_t stopping times taking their values in $[0, T]$.

$\mathbb{J}_{\text{BMO}}^2$ is the space of predictable and \mathcal{E} -measurable applications $U : \Omega \times [0, T] \times E$ such that

$$\|U\|_{\mathbb{J}_{\text{BMO}}^2}^2 := \left\| \int_0^\cdot \int_E U_s(x) \tilde{\mu}(dx, ds) \right\|_{\text{BMO}}^2 < +\infty.$$

$\mathbb{H}_{\text{BMO}}^2$ is the space of \mathbb{R}^d -valued and \mathcal{F}_t -progressively measurable processes Z such that

$$\|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 := \left\| \int_0^\cdot Z_s dB_s \right\|_{\text{BMO}}^2 < +\infty.$$

As soon as the process $\langle M \rangle$ is defined for a martingale M , which is the case if for instance M is locally square integrable, then it is easy to see that $M \in \text{BMO}$ if the jumps of M are uniformly bounded in t by some positive constant C and

$$\sup_{\tau \in \mathcal{T}_0^T} \left\| \mathbb{E}_\tau^\mathbb{P} [\langle M \rangle_T - \langle M \rangle_\tau] \right\|_\infty \leq C.$$

Furthermore the BMO norm of M is then smaller than $2C$.

We also recall the so called energy inequalities (see [17] and the references therein). Let $Z \in \mathbb{H}_{\text{BMO}}^2$, $U \in \mathbb{J}_{\text{BMO}}^2$ and $p \geq 1$. Then we have

$$\mathbb{E}^\mathbb{P} \left[\left(\int_0^T |Z_s|^2 ds \right)^p \right] \leq 2p! \left(4 \|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 \right)^p \quad (2.3)$$

$$\mathbb{E}^\mathbb{P} \left[\left(\int_0^T \int_E U_s^2(x) \nu_s(dx) ds \right)^p \right] \leq 2p! \left(4 \|U\|_{\mathbb{J}_{\text{BMO}}^2}^2 \right)^p. \quad (2.4)$$

Let us now turn to more precise properties and estimates for BMO martingales. It is a classical result (see [17]) that the Doléans-Dade exponential of a continuous BMO martingale is a uniformly integrable martingale. Things become a bit more complicated in the càdlàg case, and more assumptions are needed. Let us first define the Doléans-Dade exponential of a square integrable martingale X , denoted $\mathcal{E}(x)$. This is as usual (see for instance Protter [33]) the unique solution Z of the SDE

$$Z_t = 1 + \int_0^t Z_{s-} dX_s, \quad \mathbb{P} - a.s.,$$

and is given by the formula

$$\mathcal{E}(X)_t = e^{X_t - \frac{1}{2} \langle X^c \rangle_t} \prod_{0 < s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

One of the first results concerning Doléans-Dade exponential of BMO martingales was proved by Doléans-Dade and Meyer [10]. They showed that

Proposition 2.1. *Let M be a càdlàg BMO martingale such that $\|M\|_{\text{BMO}} < 1/8$. Then $\mathcal{E}(M)$ is a strictly positive uniformly integrable martingale.*

The constraint on the norm of the martingale being rather limiting for applications, this result was subsequently improved by Kazamaki [15], where the constraints is now on the jumps of the martingale

Proposition 2.2. *Let M be a càdlàg BMO martingale such that there exists $\delta > 0$ with $\Delta M_t \geq -1 + \delta$, for all $t \in [0, T]$, $\mathbb{P} - a.s.$ Then $\mathcal{E}(M)$ is a strictly positive uniformly integrable martingale.*

Furthermore, we emphasize, as recalled in the counter-example of Remark 2.3 in [17], that a complete generalization to the càdlàg case is not possible. We also refer the reader to Lépingle and Mémin [22] and [23] for general sufficient conditions for the uniform integrability of Doléans-Dade exponentials of càdlàg martingales. This also allows us to obtain immediately a Girsanov Theorem in this setting, which will be extremely useful throughout the paper.

Proposition 2.3. *Let us consider the following càdlàg martingale M*

$$M_t := \int_0^t \varphi_s dB_s + \int_0^t \int_E \gamma_s(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s.,$$

where $(\varphi, \gamma) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ and where there exists $\delta > 0$ with $\gamma_t \geq -1 + \delta$, $\mathbb{P} \times d\nu_t - a.e.$, for all $t \in [0, T]$.

Then, the probability measure \mathbb{Q} defined by $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}(M)$, is indeed well-defined and starting from any \mathbb{P} -martingale, by, as usual, changing adequately the drift and the jump intensity, we can obtain a \mathbb{Q} -martingale.

We now address the question of the so-called reverse Hölder inequality, which implies in the continuous case that if M is a BMO martingale, there exists some $r > 1$ such that $\mathcal{E}(M)$ is L^r -integrable. As for the previous result on uniform integrability, this was extended to the càdlàg case first in [10] and [16], with the additional assumption that the BMO norm or the jumps of M are sufficiently small. The following generalization is taken from [13]

Proposition 2.4. *Let M be a càdlàg BMO martingale such that there exists $\delta > 0$ with $\Delta M_t \geq -1 + \delta$, for all $t \in [0, T]$, $\mathbb{P} - a.s.$ Then $\mathcal{E}(M)$ is in L^r for some $r > 1$.*

We finish this short survey by considering the so-called Muckenhoupt condition. In [17], Kazamaki showed that the BMO property of a continuous martingale M was equivalent to the existence of some $p > 1$ such that $\mathbb{P} - a.s.$

$$\sup_{0 \leq t \leq T} \mathbb{E}_t^{\mathbb{P}} \left[\left(\frac{\mathcal{E}(M)_t}{\mathcal{E}(M)_T} \right)^{\frac{1}{p-1}} \right] \leq C_p, \quad (2.5)$$

for some constant C_p depending only on p . The generalization to the càdlàg case has once more been obtained by Kazamaki [16] who showed that

Proposition 2.5. *Let M be a càdlàg martingale. Then*

- (i) *If $-1 < \Delta M_t \leq C$ for some constant C and for every t , and if M satisfies (2.5) for some $p > 1$, then M is BMO.*
- (ii) *If M is BMO, then there exists some $a > 0$ such that $-1 < a\Delta M_t \leq C_a$ and such that aM satisfies (2.5) for some $p > 1$.*

2.4 The non-linear generator

Recall the Definition 1.1 of backward SDEs with jumps. We need now to specify in more details the assumptions we make on the generator g . The most important assumption in our setting in our setting will be the quadratic assumption of Assumption 2.1 (ii) below. It is the natural generalization to the jump case of the usual quadratic growth assumption in z . Before proceeding further, let us define the following function

$$j_t(u) := \int_E \left(e^{u(x)} - 1 - u(x) \right) \nu_t(dx).$$

This function $j(u)$ plays the same role for the u variable as the square function for the variable z . In order to understand this, let us consider the following "simplest" quadratic BSDE with jumps

$$y_t = \xi + \int_t^T \left(\frac{\gamma}{2} |z_s|^2 + \frac{1}{\gamma} j_s(\gamma u_s) \right) ds - \int_t^T z_s dB_s - \int_t^T \int_E u_s(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s.$$

Then a simple application of Itô's formula gives formally

$$e^{\gamma y_t} = e^{\gamma \xi} - \int_t^T e^{\gamma y_s} z_s dB_s - \int_t^T \int_E e^{\gamma y_s} \left(e^{\gamma u_s(x)} - 1 \right) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s.$$

Still formally, taking the conditional expectation above gives finally

$$y_t = \frac{1}{\gamma} \ln \left(\mathbb{E}_t^{\mathbb{P}} \left[e^{\gamma \xi} \right] \right), \quad \mathbb{P} - a.s.,$$

and we recover the so-called entropic risk measure which in the continuous case corresponds to a BSDE with generator $\frac{\gamma}{2} |z|^2$.

Of course, for the above to make sense, the function j must at the very least be well defined. A simple application of Taylor's inequalities shows that if the function $x \mapsto u(x)$ is bounded $d\nu_t - a.e.$ for every $0 \leq t \leq T$, then we have for some constant $C > 0$

$$0 \leq e^{u(x)} - 1 - u(x) \leq C u^2(x), \quad d\nu_t - a.e. \text{ for every } 0 \leq t \leq T.$$

Hence, if we introduce for $1 < p \leq +\infty$ the spaces

$$L^p(\nu) := \{u, \mathcal{E}\text{-mesurable, such that } u \in L^p(\nu_t) \text{ for all } 0 \leq t \leq T\},$$

then j is well defined on $L^2(\nu) \cap L^\infty(\nu)$.

We now give our quadratic growth assumption on the generator g

Assumption 2.1. *[Quadratic growth]*

(i) For fixed (y, z, u) , g is \mathbb{F} -progressively measurable.

(ii) For any $p \geq 1$

$$\text{ess sup}_{\tau \in \mathcal{T}_0^T}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}} \left[\left(\int_\tau^T |g_t(0, 0, 0)| dt \right)^p \right] < +\infty, \quad \mathbb{P} - a.s. \quad (2.6)$$

(iii) g has the following growth property. There exists $(\beta, \gamma) \in \mathbb{R}_+ \times \mathbb{R}_+^*$ and a positive predictable process α satisfying the same integrability condition (2.6) as $g_t(0, 0, 0)$, such that for all (ω, t, y, z, u)

$$-\alpha_t - \beta |y| - \frac{\gamma}{2} |z|^2 - \frac{1}{\gamma} j_t(-\gamma u) \leq g_t(\omega, y, z, u) - g_t(0, 0, 0) \leq \alpha_t + \beta |y| + \frac{\gamma}{2} |z|^2 + \frac{1}{\gamma} j_t(\gamma u). \quad (2.7)$$

Remark 2.1. We emphasize that unlike the usual quadratic growth assumptions for continuous BSDEs, condition (2.7) is not symmetric. It is mainly due to the fact that unlike the functions $|\cdot|$ and $|\cdot|^2$, the function j is not even. Moreover, by having this non-symmetric condition, it is easily seen that if Y is a solution to equation (2.2) with a generator satisfying the condition (2.7), then $-Y$ is also a solution to a BSDE whose generator satisfy the same condition (2.7). More precisely, if (Y, Z, U) solves equation (2.2), then $(-Y, -Z, -U)$ solves the BSDEJ with terminal condition $-\xi$ and generator $\tilde{g}_t(y, z, u) := -g_t(-y, -z, -u)$ which clearly also satisfies (2.7). This will be important for the proof of Lemma 2.1.

We also want to insist on the structure which appears in (2.7). Indeed, the constant γ in front of the quadratic term in z is the same as the one appearing in the term involving the function j . As already seen for the entropic risk-measure above, if the constants had been different, say respectively γ_1 and γ_2 , the exponential transformation would have failed. Moreover, since the function $\gamma \mapsto \gamma^{-1}j_t(\gamma u)$ is not monotone, then we cannot increase or decrease γ_1 and γ_2 to recover the desired estimate (2.7).

2.5 A priori estimates

We first prove a first result, which shows a link between the BMO spaces and quadratic BSDEs with jumps. We emphasize that only Assumption 2.1 is necessary to obtain them. Before proceeding, we denote for all $t \in [0, T]$, \mathcal{T}_t^T the collection of stopping times taking values in $[t, T]$. We also define for every $x \in \mathbb{R}$ and every $\eta \neq 0$

$$h_\eta(x) := \frac{e^{\eta x} - 1 - \eta x}{\eta}.$$

The function h_η already appears in our growth Assumption 2.1(ii), and the following trivial property that it satisfies is going to be crucial for us

$$h_{2\eta}(x) = \frac{(e^{\eta x} - 1)^2}{2\eta} + h_\eta(x). \quad (2.8)$$

We also give the two following inequalities which are of the utmost importance in our jump setting. We emphasize that the first one is trivial, while the second one can be proved using simple but tedious algebra.

$$2 \leq e^x + e^{-x}, \text{ for all } x \in \mathbb{R} \quad (2.9)$$

$$x^2 \leq a(e^x - 1)^2 + \frac{(1 - e^{-x})^2}{a}, \text{ for all } (a, x) \in \mathbb{R}_+^* \times \mathbb{R}. \quad (2.10)$$

We then have

Lemma 2.1. *Let Assumption 2.1 hold. Assume that (Y, Z, U) is a solution of the BSDE (2.2) such that $(Z, U) \in \mathcal{Z} \times \mathcal{U}$, the jumps of Y are bounded and*

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T}^{\mathbb{P}} \mathbb{E}_\tau^{\mathbb{P}} \left[\exp \left(2\gamma \sup_{\tau \leq t \leq T} \pm Y_t \right) \vee \exp \left(4\gamma \sup_{\tau \leq t \leq T} \pm Y_t \right) \right] < +\infty, \quad \mathbb{P} - a.s. \quad (2.11)$$

Then $Z \in \mathbb{H}_{\text{BMO}}^2$ and $U \in \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$.

Proof. First of all, since the size of the jumps of Y is bounded, there exists a version of U , that is to say that there exists a predictable function \tilde{U} such that for all $t \in [0, T]$

$$\int_E \left| \tilde{U}_t(x) - U_t(x) \right|^2 \nu_t(dx) = 0,$$

and such that

$$\left| \tilde{U}_t(x) \right| \leq C, \text{ for all } t.$$

For the sake of simplicity, we will always consider this version and we still denote it U . For the proof of this result, we refer to Morlais [28].

Let us consider the following processes

$$\int_0^T e^{2\gamma Y_t} Z_t dB_t \text{ and } \int_0^T e^{2\gamma Y_{t-}} \left(e^{2\gamma U_t(x)} - 1 \right) \tilde{\mu}(dx, dt).$$

We will first show that they are local martingales. Indeed, we have

$$\int_0^T e^{4\gamma Y_t} Z_t^2 dt \leq \exp \left(4\gamma \sup_{0 \leq t \leq T} Y_t \right) \int_0^T Z_t^2 dt < +\infty, \mathbb{P} - a.s.,$$

since $Z \in \mathcal{Z}$ and (2.11) holds.

Similarly, we have

$$\int_0^T \int_E e^{4\gamma Y_t} U_t^2(x) \nu_t(dx) dt \leq \exp \left(4\gamma \sup_{0 \leq t \leq T} Y_t \right) \int_0^T \int_E U_t^2(x) \nu_t(dx) dt < +\infty, \mathbb{P} - a.s.,$$

since $U \in \mathcal{U}$ and (2.11) holds.

Let now $(\tau_n)_{n \geq 1}$ be a localizing sequence for the \mathbb{P} -local martingales above. By Itô's formula under \mathbb{P} applied to $e^{2\gamma Y_t}$, we have for every $\tau \in \mathcal{T}_0^T$

$$\begin{aligned} & \frac{4\gamma^2}{2} \int_\tau^{\tau_n} e^{2\gamma Y_t} |Z_t|^2 dt + 2\gamma \int_\tau^{\tau_n} \int_E e^{2\gamma Y_t} h_{2\gamma}(U_t(x)) \nu_t(dx) dt \\ &= e^{2\gamma Y_{\tau_n}} - e^{-2\gamma Y_\tau} + 2\gamma \int_\tau^{\tau_n} e^{2\gamma Y_t} g_t(Y_t, Z_t, U_t) dt - 2\gamma \int_\tau^{\tau_n} e^{2\gamma Y_t} Z_t dB_t \\ & \quad - \int_\tau^{\tau_n} \int_E e^{2\gamma Y_{t-}} \left(e^{2\gamma U_t(x)} - 1 \right) \tilde{\mu}(dx, dt) \\ &\leq e^{2\gamma Y_{\tau_n}} - e^{-2\gamma Y_\tau} + 2\gamma \int_\tau^{\tau_n} e^{2\gamma Y_t} \left(\alpha_t + |g_t(0, 0, 0)| + \beta |Y_t| + \frac{\gamma}{2} |Z_t|^2 + \int_E h_\gamma(U_t(x)) \nu_t(dx) \right) dt \\ & \quad - 2\gamma \int_\tau^{\tau_n} e^{2\gamma Y_t} Z_t dB_t - \int_\tau^{\tau_n} \int_E e^{2\gamma Y_{t-}} \left(e^{2\gamma U_t(x)} - 1 \right) \tilde{\mu}(dx, dt), \mathbb{P} - a.s. \end{aligned}$$

Now the situation is going to be different from the continuous case, and the property (2.8) is going to be important. Indeed, we can take conditional expectation and thus obtain

$$\begin{aligned} & \mathbb{E}_\tau^\mathbb{P} \left[\gamma^2 \int_\tau^{\tau_n} e^{2\gamma Y_t} |Z_t|^2 dt + \int_\tau^{\tau_n} \int_E e^{2\gamma Y_t} \left(e^{\gamma U_t(x)} - 1 \right)^2 \nu_t(dx) dt \right] \\ &\leq C \left(1 + \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^{\tau_n} (\alpha_t + |g_t(0, 0, 0)|) dt \right)^2 + \exp \left(2\gamma \sup_{\tau \leq t \leq T} Y_t \right) + \exp \left(4\gamma \sup_{\tau \leq t \leq T} Y_t \right) \right] \right) \\ &\leq C \left(1 + \mathbb{E}_\tau^\mathbb{P} \left[\exp \left(2\gamma \sup_{\tau \leq t \leq T} Y_t \right) \vee \exp \left(4\gamma \sup_{\tau \leq t \leq T} Y_t \right) \right] \right), \end{aligned}$$

where we used the inequality $2ab \leq a^2 + b^2$, the fact that for all $x \in \mathbb{R}$, $|x|e^x \leq C(1 + e^{2x})$ for some constant $C > 0$ (which as usual can change value from line to line) and the fact that Assumption 2.1(ii) and (iii) hold.

Using Fatou's lemma and the monotone convergence Theorem, we obtain

$$\begin{aligned} & \mathbb{E}_\tau^\mathbb{P} \left[\gamma^2 \int_\tau^T e^{2\gamma Y_t} |Z_t|^2 dt + \int_\tau^T \int_E e^{2\gamma Y_t} \left(e^{\gamma U_t(x)} - 1 \right)^2 \nu_t(dx) dt \right] \\ & \leq C \left(1 + \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T}^\mathbb{P} \mathbb{E}_\tau^\mathbb{P} \left[\exp \left(2\gamma \sup_{\tau \leq t \leq T} Y_t \right) \vee \exp \left(4\gamma \sup_{\tau \leq t \leq T} Y_t \right) \right] \right). \end{aligned} \quad (2.12)$$

Now, we apply the above estimate for the solution $(-Y, -Z, -U)$ of the BSDEJ with terminal condition $-\xi$ and generator $\tilde{g}_t(y, z, u) := -g_t(-y, -z, -u)$, which still satisfies Assumption 2.1 (see Remark 2.1)

$$\begin{aligned} & \mathbb{E}_\tau^\mathbb{P} \left[\gamma^2 \int_\tau^T e^{-2\gamma Y_t} |Z_t|^2 dt + \int_\tau^T \int_E e^{-2\gamma Y_t} \left(e^{-\gamma U_t(x)} - 1 \right)^2 \nu_t(dx) dt \right] \\ & \leq C \left(1 + \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T}^\mathbb{P} \mathbb{E}_\tau^\mathbb{P} \left[\exp \left(2\gamma \sup_{\tau \leq t \leq T} (-Y_t) \right) \vee \exp \left(4\gamma \sup_{\tau \leq t \leq T} (-Y_t) \right) \right] \right). \end{aligned} \quad (2.13)$$

Let us now sum the inequalities (2.12) and (2.13). We obtain

$$\begin{aligned} & \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T \gamma^2 (e^{2\gamma Y_t} + e^{-2\gamma Y_t}) |Z_t|^2 + \int_E e^{2\gamma Y_t} \left(e^{\gamma U_t(x)} - 1 \right)^2 + e^{-2\gamma Y_t} \left(e^{-\gamma U_t(x)} - 1 \right)^2 \nu_t(dx) dt \right] \\ & \leq C \left(1 + \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T}^\mathbb{P} \mathbb{E}_\tau^\mathbb{P} \left[\sup_{\tau \leq t \leq T} \{ \exp(2\gamma Y_t) \vee \exp(4\gamma Y_t) + \exp(2\gamma(-Y_t)) \vee \exp(4\gamma(-Y_t)) \} \right] \right). \end{aligned}$$

Finally, from the inequalities (2.9) and (2.10), this shows the desired result. \square

Remark 2.2. *In the above Proposition, if we only assume that*

$$\mathbb{E}^\mathbb{P} \left[\exp \left(2\gamma \sup_{0 \leq t \leq T} \pm Y_t \right) \vee \exp \left(4\gamma \sup_{0 \leq t \leq T} \pm Y_t \right) \right] < +\infty,$$

then the exact same proof would show that $(Z, U) \in \mathbb{H}^2 \times \mathbb{J}^2$. Moreover, using the Neveu-Garsia Lemma in the same spirit as [4], we could also show that $(Z, U) \in \mathbb{H}^p \times \mathbb{J}^p$ for all $p > 1$.

We emphasize that the results of this Proposition highlight the fact that we do not necessarily need to consider solutions with a bounded Y in the quadratic case to obtain *a priori* estimates. It is enough to assume the existence of some exponential moments. This generalizes to the jump case some of the ideas developed in [7] and [4]. Nonetheless, our proof of existence will rely heavily on BMO properties of the solution, and the simplest condition to obtain the estimate (2.11) is to assume that Y is indeed bounded. The aim of the following Proposition is to show that we can control the \mathcal{S}^∞ norm of Y by the L^∞ norm of ξ . Since the proof is very similar to the proof of Lemma 1 in [7], we will omit it.

Proposition 2.6. *Let $\xi \in \mathbb{L}^\infty$. Let Assumption 2.1 hold and assume that $|g(0, 0, 0)| + \alpha \leq M$ for some constant $M > 0$. Let $(Y, Z, U) \in \mathcal{S}^\infty \times \mathbb{H}^2 \times \mathbb{J}^2$ be a solution of the BSDE (2.2). Then we have*

$$|Y_t| \leq \gamma M \frac{e^{\beta(T-t)} - 1}{\beta} + \gamma e^{\beta(T-t)} \|\xi\|_{\mathbb{L}^\infty}.$$

2.6 Existence and uniqueness for a small terminal condition

The aim of this Section is to obtain an existence and uniqueness result for BSDEJ with quadratic growth when the terminal condition is small enough. However, we will need more assumptions for our proof to work. First, we assume from now on that we have the following martingale representation property. We need this assumption since we will rely on the existence results in [1] or [24] which need the martingale representation.

Assumption 2.2. *Any local martingale M has the predictable representation property, that is to say, that there exists a unique predictable process H and a unique predictable function U such that $(H, U) \in \mathcal{Z} \times \mathcal{U}$ and*

$$M_t = M_0 + \int_0^t H_s dB_s + \int_0^t \int_E U_s(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s.$$

Remark 2.3. *This martingale representation property holds for instance when the compensator ν does not depend on ω , i.e when ν is the compensator of the counting measure of an additive process in the sense of Sato [35]. It also holds when ν has the particular form described in [18], in which case ν depends on ω .*

Of course, we also need to assume more properties for our generator g .

Assumption 2.3. *[Lipschitz assumption]*

Let Assumption 2.1(i),(ii) hold and assume furthermore that

(i) *g is uniformly Lipschitz in y .*

$$|g_t(\omega, y, z, u) - g_t(\omega, y', z, u)| \leq C |y - y'| \quad \text{for all } (\omega, t, y, y', z, u).$$

(ii) $\exists \mu > 0$ and $\phi \in \mathbb{H}_{\text{BMO}}^2$ such that for all (t, y, z, z', u)

$$|g_t(\omega, y, z, u) - g_t(\omega, y, z', u) - \phi_t \cdot (z - z')| \leq \mu |z - z'| (|z| + |z'|).$$

(iii) $\exists \mu > 0$ and $\psi \in \mathbb{J}_{\text{BMO}}^2$ such that for all (t, x)

$$C_1(1 \wedge |x|) \leq \psi_t(x) \leq C_2(1 \wedge |x|),$$

where $C_2 > 0$, $C_1 \geq -1 + \delta$ where $\delta > 0$. Moreover, for all (ω, t, y, z, u, u')

$$|g_t(\omega, y, z, u) - g_t(\omega, y, z, u') - \psi_t \cdot (u - u')| \leq \mu \|u - u'\|_{L^2(\nu_t)} \left(\|u\|_{L^2(\nu_t)} + \|u'\|_{L^2(\nu_t)} \right),$$

where $\langle u_1, u_2 \rangle_t := \int_E u_1(x) u_2(x) \nu_t(dx)$ is the scalar product in $L^2(\nu_t)$.

Remark 2.4. *Let us comment on the above assumptions. The first one concerning Lipschitz continuity in the variable y is classical in the BSDE theory. The two others may seem a bit complicated, but they are almost equivalent to saying that the function g is locally Lipschitz in z and u . In the case of the variable z for instance, those two properties would be equivalent if the process ϕ were bounded. Here we allow something a bit more general by letting ϕ be unbounded but in $\mathbb{H}_{\text{BMO}}^2$. Once again, since these assumptions allow us to apply the Girsanov property of Proposition 2.3, we do not need to bound the processes and BMO type conditions are sufficient. Moreover, Assumption 2.3 also implies a weaker version of Assumption 2.1. Indeed, it implies clearly that*

$$|g_t(y, z, u) - g_t(0, 0, 0) - \phi_t \cdot z - \psi_t \cdot u| \leq C |y| + \mu \left(|z|^2 + \|u\|_{L^2(\nu_t)}^2 \right).$$

Then, for any $u \in L^2(\nu) \cap L^\infty(\nu)$ and for any $\gamma > 0$, we have using the mean value Theorem

$$\frac{\gamma}{2} e^{-\gamma \|u\|_{L^\infty(\nu)}} \|u\|_{L^2(\nu_t)}^2 \leq \frac{1}{\gamma} j_t(\pm \gamma u) \leq \frac{\gamma}{2} e^{\gamma \|u\|_{L^\infty(\nu)}} \|u\|_{L^2(\nu_t)}^2.$$

Therefore we deduce using the Cauchy-Schwarz inequality and the trivial inequality $2ab \leq a^2 + b^2$

$$\begin{aligned} g_t(y, z, u) - g_t(0, 0, 0) &\leq \frac{|\phi_t|^2}{2} + \frac{\|\psi_t\|_{L^2(\nu_t)}^2}{2} + C|y| + \left(\mu + \frac{1}{2}\right) \left(|z|^2 + \frac{2e^{\gamma \|u\|_{L^\infty(\nu)}}}{\gamma^2} j_t(\gamma u)\right) \\ g_t(y, z, u) - g_t(0, 0, 0) &\geq -\frac{|\phi_t|^2}{2} - \frac{\|\psi_t\|_{L^2(\nu_t)}^2}{2} - C|y| - \left(\mu + \frac{1}{2}\right) \left(|z|^2 - \frac{2e^{\gamma \|u\|_{L^\infty(\nu)}}}{\gamma^2} j_t(-\gamma u)\right). \end{aligned}$$

We now check that the term $|\phi_t|^2/2 + \|\psi_t\|_{L^2(\nu_t)}^2/2$ above satisfies the integrability condition (2.6).

We have for any $p \geq 1$

$$\begin{aligned} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^T \left(\frac{|\phi_t|^2}{2} + \frac{\|\psi_t\|_{L^2(\nu_t)}^2}{2} \right) dt \right)^p \right] &\leq C_p \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^T |\phi_t|^2 dt \right)^p \right] \\ &\quad + C_p \operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^T \|\psi_t\|_{L^2(\nu_t)}^2 dt \right)^p \right], \end{aligned}$$

for some constant $C_p > 0$ depending only on p .

Let us concentrate only on the first term on the right-hand side above, since the other one can be treated similarly. Given that our family of stopping times is upward directed, we know by definition of the essential supremum that there exists some sequence $(\tau_n)_{n \geq 0} \subset \mathcal{T}_0^T$ such that

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^T |\phi_t|^2 dt \right)^p \right] = \lim_{n \rightarrow +\infty} \uparrow \mathbb{E}_{\tau_n}^\mathbb{P} \left[\left(\int_{\tau_n}^T |\phi_t|^2 dt \right)^p \right], \quad \mathbb{P} - a.s.$$

Thus, by the monotone convergence theorem and the tower property of conditional expectation, we have

$$\begin{aligned} \mathbb{E}^\mathbb{P} \left[\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^T |\phi_t|^2 dt \right)^p \right] \right] &= \lim_{n \rightarrow +\infty} \uparrow \mathbb{E}^\mathbb{P} \left[\mathbb{E}_{\tau_n}^\mathbb{P} \left[\left(\int_{\tau_n}^T |\phi_t|^2 dt \right)^p \right] \right] \\ &= \lim_{n \rightarrow +\infty} \uparrow \mathbb{E}^\mathbb{P} \left[\left(\int_{\tau_n}^T |\phi_t|^2 dt \right)^p \right] \\ &\leq \mathbb{E}^\mathbb{P} \left[\left(\int_0^T |\phi_t|^2 dt \right)^p \right] < +\infty, \end{aligned}$$

by the energy inequalities (2.3), which hold since $\phi \in \mathbb{H}_{\text{BMO}}^2$.

Consequently, we have

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_0^T} \mathbb{E}_\tau^\mathbb{P} \left[\left(\int_\tau^T |\phi_t|^2 dt \right)^p \right] < +\infty, \quad \mathbb{P} - a.s.,$$

and a similar inequality holds for ψ .

Hence, we have obtained a growth property which is similar to (2.7), the only difference being that the constants appearing in the quadratic term in z and the term involving the function j are not the same. This prevents us from recovering the structure already mentioned in Remark 2.1.

We now show that if we can solve the BSDEJ (2.2) for a generator g satisfying Assumption 2.3 with $\phi = 0$ and $\psi = 0$, we can immediately obtain the existence for general ϕ and ψ . This will simplify our subsequent proof of existence. Notice that the result relies essentially on the Girsanov Theorem of Proposition 2.3.

Lemma 2.2. *Define*

$$\bar{g}_t(\omega, y, z, u) := g_t(\omega, y, z, u) - \langle \phi_t(\omega), z \rangle_{\mathbb{R}^d} - \langle \psi_t(\omega), u \rangle_{L^2(\nu_t)}.$$

Then (Y, Z, U) is a solution of the BSDEJ with generator g and terminal condition ξ under \mathbb{P} if and only if (Y, Z, U) is a solution of the BSDEJ with generator \bar{g} and terminal condition ξ under \mathbb{Q} where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E} \left(\int_0^T \phi_s dB_s + \int_0^T \int_E \psi_s(x) \tilde{\mu}(dx, ds) \right).$$

Proof. We have clearly

$$\begin{aligned} Y_t &= \xi + \int_t^T g_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds) \\ \Leftrightarrow Y_t &= \xi + \int_t^T \bar{g}_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s (dB_s - \phi_s ds) \\ &\quad - \int_t^T \int_E U_s(x) (\tilde{\mu}(dx, ds) - \psi_s(x) \nu_s(dx) ds). \end{aligned}$$

Now, by our BMO assumptions on ϕ and ψ and the fact that we assumed that $\psi \geq -1 + \delta$, we can apply Proposition 2.3 and \mathbb{Q} is well defined. Then by Girsanov Theorem, we know that $dB_s - \phi_s ds$ and $\tilde{\mu}(dx, ds) - \psi_s(x) \nu_s(dx) ds$ are martingales under \mathbb{Q} . Hence the desired result. \square

Remark 2.5. *It is clear that if g satisfies Assumption 2.3, then \bar{g} defined above satisfies Assumption 2.3 with $\phi = \psi = 0$.*

Following Lemma 2.2 we assume for the time being that $g(0, 0, 0) = \phi = \psi = 0$. Our first result is the following

Theorem 2.1. *Assume that*

$$\|\xi\|_\infty \leq \frac{1}{2\sqrt{15}\sqrt{2670}\mu e^{\frac{3}{2}CT}},$$

where C is the Lipschitz constant of g in y , and μ is the constant appearing in Assumption 2.3. Then under Assumption 2.3 with $\phi = 0$, $\psi = 0$ and $g(0, 0, 0) = 0$, there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ of the BSDEJ (2.2).

Remark 2.6. *Notice that in the above Theorem, we do not need Assumption 2.1(iii) to hold. This is linked to the fact that, as discussed in Remark 2.4, Assumption 2.3 implies a weak version of Assumption 2.1(iii), which is sufficient for our purpose here.*

Proof. We first recall that we have with Assumption 2.3 when $g(0, 0, 0) = \phi = \psi = 0$

$$|g_t(y, z, u)| \leq C|y| + \mu|z|^2 + \mu\|u\|_{L^2(\nu_t)}^2. \quad (2.14)$$

Consider now the map $\Phi : (y, z, u) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu) \rightarrow (Y, Z, U)$ defined by

$$Y_t = \xi + \int_t^T g_s(Y_s, z_s, u_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds). \quad (2.15)$$

The above is nothing more than a BSDE with jumps whose generator depends only on Y and is Lipschitz. Besides, since $(z, u) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$, using (2.6), (2.14) and the energy inequalities (2.3) we clearly have

$$\mathbb{E}^\mathbb{P} \left[\left(\int_0^T |g_s(0, z_s, u_s)| ds \right)^2 \right] < +\infty.$$

Hence, the existence of $(Y, Z, U) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ is ensured by the results of Barles, Buckdahn and Pardoux [1] or Li and Tang [24] for Lipschitz BSDEs with jumps. Of course, we could have let the generator in (2.15) depend on (y_s, z_s, u_s) instead. The existence of (Y, Z, U) would then have been a consequence of the predictable martingale representation Theorem. However, the form that we have chosen will simplify some of the following estimates.

Step 1: We first show that $(Y, Z, U) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$.

Recall that by the Lipschitz hypothesis in y , there exists a bounded process λ such that

$$g_s(Y_s, z_s, u_s) = \lambda_s Y_s + g_s(0, z_s, u_s).$$

Let us now apply Itô's formula to $e^{\int_t^s \lambda_u du} |Y_s|$. We obtain easily from Assumption 2.3

$$\begin{aligned} Y_t &= \mathbb{E}_t^\mathbb{P} \left[e^{\int_t^T \lambda_s ds} \xi + \int_t^T e^{\int_t^s \lambda_u du} (\lambda_s Y_s + g_s(0, z_s, u_s)) ds - \int_t^T \lambda_s e^{\int_t^s \lambda_u du} Y_s ds \right] \\ &\leq \mathbb{E}_t^\mathbb{P} \left[e^{\int_t^T \lambda_s ds} \xi + \mu \int_t^T e^{\int_t^s \lambda_u du} \left(|z_s|^2 + \int_E u_s^2(x) \nu_s(dx) \right) ds \right] \\ &\leq \|\xi\|_\infty + C \left(\|z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|u\|_{\mathbb{J}_{\text{BMO}}^2}^2 \right). \end{aligned}$$

Therefore Y is bounded and consequently, since its jumps are also bounded, we know that there is a version of U such that

$$\|U\|_{L^\infty(\nu)} \leq 2 \|Y\|_{\mathcal{S}^\infty}.$$

Let us now prove that $(Z, U) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2$. Applying Itô's formula to $e^{\eta t} |Y_t|^2$ for some $\eta > 0$, we obtain for any stopping time $\tau \in \mathcal{T}_0^T$

$$\begin{aligned} e^{\eta \tau} |Y_\tau|^2 &+ \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T e^{\eta s} |Z_s|^2 ds + \int_\tau^T \int_E e^{\eta s} U_s^2(x) \nu_s(dx) ds \right] \\ &= \mathbb{E}_\tau^\mathbb{P} \left[e^{\eta T} \xi^2 + 2 \int_\tau^T e^{\eta s} Y_s g_s(Y_s, z_s, u_s) ds - \eta \int_\tau^T e^{\eta s} |Y_s|^2 ds \right] \\ &\leq \mathbb{E}_\tau^\mathbb{P} \left[e^{\eta T} \xi^2 + (2C - \eta) \int_\tau^T e^{\eta s} |Y_s|^2 ds + 2 \|Y\|_{\mathcal{S}^\infty} \int_\tau^T e^{\eta s} |g_s(0, z_s, u_s)| ds \right]. \end{aligned}$$

Choosing $\eta = 2C$, and using the elementary inequality $2ab \leq \frac{a^2}{\varepsilon} + \varepsilon b^2$, we obtain

$$\begin{aligned} |Y_\tau|^2 &+ \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T |Z_s|^2 ds + \int_\tau^T \int_E U_s^2(x) \nu_s(dx) ds \right] \\ &\leq \mathbb{E}_\tau^\mathbb{P} \left[e^{\eta T} \xi^2 + \varepsilon \|Y\|_{\mathcal{S}^\infty}^2 + \frac{e^{2\eta T}}{\varepsilon} \left(\int_\tau^T |g_s(0, z_s, u_s)| ds \right)^2 \right]. \end{aligned}$$

Hence,

$$(1 - \varepsilon) \|Y\|_{\mathcal{S}^\infty}^2 + \|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|U\|_{\mathbb{J}_{\text{BMO}}^2}^2 \leq e^{\eta T} \|\xi\|_\infty^2 + 64\mu^2 \frac{e^{2\eta T}}{\varepsilon} \left(\|z\|_{\mathbb{H}_{\text{BMO}}^2}^4 + \|u\|_{\mathbb{J}_{\text{BMO}}^2}^4 \right).$$

And finally, choosing $\varepsilon = 1/2$

$$\|Y\|_{\mathcal{S}^\infty}^2 + \|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|U\|_{\mathbb{J}_{\text{BMO}}^2}^2 \leq 2e^{\eta T} \|\xi\|_\infty^2 + 256\mu^2 e^{2\eta T} \left(\|z\|_{\mathbb{H}_{\text{BMO}}^2}^4 + \|u\|_{\mathbb{J}_{\text{BMO}}^2}^4 \right).$$

Our problem now is that the norms for Z and U in the left-hand side above are to the power 2, while they are to the power 4 on the right-hand side. Therefore, it will clearly be impossible for us to prevent an explosion if we do not first start by restricting ourselves in some ball with a well chosen radius. This is exactly what we are going to do. Define therefore $R = \frac{1}{2\sqrt{2670}\mu e^{\nu T}}$, and assume that $\|\xi\|_\infty \leq \frac{R}{\sqrt{15}e^{\frac{1}{2}\eta T}}$ and that

$$\|y\|_{\mathcal{S}^\infty}^2 + \|z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|u\|_{\mathbb{J}_{\text{BMO}}^2}^2 + \|u\|_{L^\infty(\nu)}^2 \leq R^2.$$

Denote $\Lambda := \|Y\|_{\mathcal{S}^\infty}^2 + \|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|U\|_{\mathbb{J}_{\text{BMO}}^2}^2 + \|U\|_{L^\infty(\nu)}^2$. We have, since $\|U\|_{L^\infty(\nu)}^2 \leq 4\|Y\|_{\mathcal{S}^\infty}^2$

$$\begin{aligned} \Lambda &\leq 5\|Y\|_{\mathcal{S}^\infty}^2 + \|Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|U\|_{\mathbb{J}_{\text{BMO}}^2}^2 \leq 10e^{\nu T} \|\xi\|_\infty^2 + 1280\mu^2 e^{2\nu T} \left(\|z\|_{\mathbb{H}_{\text{BMO}}^2}^4 + \|u\|_{\mathbb{J}_{\text{BMO}}^2}^4 \right) \\ &\leq \frac{2R^2}{3} + 3560\mu^2 e^{2\nu T} R^4 \\ &= \frac{2R^2}{3} + \frac{R^2}{3} = R^2. \end{aligned}$$

Hence if \mathcal{B}_R is the ball of radius R in $\mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$, we have shown that $\Phi(\mathcal{B}_R) \subset \mathcal{B}_R$.

Step 2: We show that Φ is a contraction in this ball of radius R .

For $i = 1, 2$ and $(y^i, z^i, u^i) \in \mathcal{B}_R$, we denote $(Y^i, Z^i, U^i) := \Phi(y^i, z^i, u^i)$ and

$$\begin{aligned} \delta y &:= y^1 - y^2, \quad \delta z := z^1 - z^2, \quad \delta u := u^1 - u^2, \quad \delta Y := Y^1 - Y^2 \\ \delta Z &:= Z^1 - Z^2, \quad \delta U := U^1 - U^2, \quad \delta g := g(Y^2, z^1, u^1) - g(Y^2, z^2, u^2). \end{aligned}$$

Arguing as above, we obtain easily

$$\|\delta Y\|_{\mathcal{S}^\infty}^2 + \|\delta Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|\delta U\|_{\mathbb{J}_{\text{BMO}}^2}^2 \leq 4e^{2\eta T} \sup_{\tau \in \mathcal{T}_0^T} \left(\mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T |\delta g_s| ds \right] \right)^2.$$

We next estimate that

$$\begin{aligned} \left(\mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T |\delta g_s| ds \right] \right)^2 &\leq 2\mu^2 \left(\mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T |\delta z_s| (|z_s^1| + |z_s^2|) ds \right] \right)^2 \\ &\quad + 2\mu^2 \left(\mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T \|\delta u_s\|_{L^2(\nu_s)} \left(\|u_s^1\|_{L^2(\nu_s)} + \|u_s^2\|_{L^2(\nu_s)} \right) ds \right] \right)^2 \\ &\leq 2\mu^2 \left(\mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T |\delta z_s|^2 ds \right] \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T (|z_s^1| + |z_s^2|)^2 ds \right] \right. \\ &\quad \left. + \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T \|\delta u_s\|_{L^2(\nu_s)}^2 ds \right] \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T \left(\|u_s^1\|_{L^2(\nu_s)} + \|u_s^2\|_{L^2(\nu_s)} \right)^2 ds \right] \right) \\ &\leq 4R^2\mu^2 \left(\mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T |\delta z_s|^2 ds \right] + \mathbb{E}_\tau^\mathbb{P} \left[\int_\tau^T \int_E \delta u_s^2(x) \nu(dx) ds \right] \right) \\ &\leq 32R^2\mu^2 \left(\|\delta z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|\delta u\|_{\mathbb{J}_{\text{BMO}}^2}^2 \right) \end{aligned}$$

From these estimates, we get

$$\begin{aligned} \|\delta Y\|_{\mathcal{S}^\infty}^2 + \|\delta Z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|\delta U\|_{\mathbb{J}_{\text{BMO}}^2}^2 + \|\delta U\|_{L^\infty(\nu)}^2 &\leq 20 \times 32R^2\mu^2 e^{2\nu T} \left(\|\delta z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|\delta u\|_{\mathbb{J}_{\text{BMO}}^2}^2 \right) \\ &= \frac{16}{267} \left(\|\delta z\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|\delta u\|_{\mathbb{J}_{\text{BMO}}^2}^2 \right). \end{aligned}$$

Therefore Φ is a contraction which has a unique fixed point. \square

Then, from Lemma 2.2, we have immediately the following corollary

Corollary 2.1. *Assume that*

$$\|\xi\|_\infty \leq \frac{1}{2\sqrt{15}\sqrt{2670}\mu e^{\frac{3}{2}CT}},$$

where C is the Lipschitz constant of g in y , and μ is the constant appearing in Assumption 2.3. Then under Assumption 2.3 with $g(0,0,0) = 0$, there exists a unique solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ of the BSDEJ (2.2).

We now show how we can get rid off the assumption that $g_t(0,0,0) = 0$.

Corollary 2.2. *Assume that*

$$\|\xi\|_\infty + D \left\| \int_0^T |g_t(0,0,0)| dt \right\|_\infty \leq \frac{1}{2\sqrt{15}\sqrt{2670}\mu e^{\frac{3}{2}CT}},$$

where C is the Lipschitz constant of g in y , μ is the constant appearing in Assumption 2.3 and D is a large enough positive constant. Then under Assumption 2.3, there exists a solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ of the BSDEJ (2.2).

Proof. By Corollary 2.1, we can show the existence of a solution to the BSDEJ with generator $\tilde{g}_t(y, z, u) := g_t(y - \int_0^t g_s(0,0,0)ds, z, u) - g_t(0,0,0)$ and terminal condition $\bar{\xi} := \xi + \int_0^T g_t(0,0,0)dt$. Indeed, even though \bar{g} is not null at $(0,0,0)$, it is not difficult to show with the same proof as in Theorem 2.1 that a solution $(\bar{Y}, \bar{Z}, \bar{U})$ exists (the same type of arguments are used in [36]). More precisely, \tilde{g} still satisfies Assumption 2.3(i) and when ϕ and ψ in Assumption 2.3 are equal to 0, we have the estimate

$$|\tilde{g}_t(y, z, u)| \leq C \left\| \int_0^T |g_s(0,0,0)| ds \right\|_\infty + C|y| + \mu|z|^2 + \mu\|u\|_{L^2(\nu_t)}^2,$$

which is the counterpart of (2.14). Thus, since the constant term in the above estimate is assumed to be small enough, it will play the same role as $\|\xi\|_\infty$ in the first Step of the proof of Theorem 2.1.

For the Step 2, everything still work thanks to the following estimate

$$\begin{aligned} |\tilde{g}_t(Y^2, z^1, u^1) - \tilde{g}_t(Y^2, z^2, u^2)| &\leq \mu|z^1 - z^2|(|z^1| + |z^2|) \\ &\quad + \mu\|u^1 - u^2\|_{L^2(\nu_t)} \left(\|u^1\|_{L^2(\nu_t)} + \|u^2\|_{L^2(\nu_t)} \right). \end{aligned}$$

Then, if we define

$$(Y_t, Z_t, U_t) := \left(\bar{Y}_t - \int_0^t g_s(0,0,0)ds, \bar{Z}_t, \bar{U}_t \right),$$

it is clear that it is a solution to the BSDEJ with generator g and terminal condition ξ . \square

Remark 2.7. *We emphasize that the above proof of existence extends readily to a terminal condition which is in \mathbb{R}^n for any $n \geq 2$.*

2.7 Existence for a bounded terminal condition

We now show that we can still prove existence of a solution for any bounded terminal condition. Nonetheless, we will need to strengthen once more our assumptions on the generator, mainly by assuming more regularity.

Assumption 2.4. (i) g is uniformly Lipschitz in y .

$$|g_t(\omega, y, z, u) - g_t(\omega, y', z, u)| \leq C |y - y'| \text{ for all } (\omega, t, y, y', z, u).$$

(ii) g is C^2 in z and there is a constant $\theta > 0$ and a process $(r_t)_{0 \leq t \leq T} \in \mathbb{H}_{\text{BMO}}^2$, such that for all (t, ω, y, z, u) ,

$$|D_z g_t(\omega, y, z, u)| \leq r_t + \theta |z|, \quad |D_{zz}^2 g_t(\omega, y, z, u)| \leq \theta.$$

(iii) g is twice Fréchet differentiable in the Banach space $L^2(\nu)$ and there are constants $\theta, \delta > 0$, $C_1 \geq -1 + \delta$, $C_2 \geq 0$ and a predictable function $m \in \mathbb{J}_{\text{BMO}}^2$ such that for all (t, ω, y, z, u, x) ,

$$|D_u g_t(\omega, y, z, u)|(x) \leq m_t(x) + \theta |u(x)|, \quad C_1(1 \wedge |x|) \leq (D_u g_t(\omega, y, z, u))(x) \leq C_2(1 \wedge |x|)$$

$$\|D_u^2 g_t(\omega, y, z, u)\|_{L^2(\nu_t)} \leq \theta.$$

Remark 2.8. The assumptions (ii) and (iii) above are generalizations to the jump case of the assumptions considered by Tevzadze [36]. They will only be useful in our proof of existence and are tailor-made to allow us to apply the Girsanov transformation of Proposition 2.3. Notice also that since the space $L^2(\nu)$ is clearly a Banach space, there is no problem to define the Fréchet derivative.

We emphasize here that Assumption 2.4 is stronger than Assumption 2.3. Indeed, we have the following result

Lemma 2.3. If Assumption 2.4(ii) and (iii) hold, then so do Assumption 2.3(ii) and (iii).

Proof. We will only show that if Assumption 2.4(iii) holds, so does Assumption 2.3(iii), the proof being similar for Assumption 2.4(ii). Since g is twice Fréchet differentiable in u , we introduce the process $\psi_t := D_u g_t(y, z, 0)$ which is bounded from above by m and from below by $C_1 \geq -1 + \delta$ by assumption. Thus, $\psi \in \mathbb{J}_{\text{BMO}}^2$. By the mean value theorem, we compute that for some $\lambda \in [0, 1]$ and with $u_\lambda := \lambda u + (1 - \lambda)u'$

$$\begin{aligned} |g_t(y, z, u) - g_t(y, z, u') - \langle \psi_t, u - u' \rangle_t| &\leq \|D_u g_t(y, z, u_\lambda) - \psi_t\| \|u - u'\|_{L^2(\nu_t)} \\ &\leq \theta \|\lambda u + (1 - \lambda)u'\|_{L^2(\nu_t)} \|u - u'\|_{L^2(\nu_t)}, \end{aligned}$$

by the bound on $D_u^2 g$. The result now follows easily. \square

We can now state our main existence result.

Theorem 2.2. Let $\xi \in \mathbb{L}^\infty$. Under Assumptions 2.1 and 2.4, there exists a solution $(Y, Z, U) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ of the BSDEJ (2.2).

The idea of the proof is to find a "good" splitting of the BSDE into the sum of BSDEs for which the terminal condition is small and existence holds. Then we paste everything together. This is during this pasting step that the regularity of the generator in z and u in Assumption 2.4 is going to be important.

Proof.

(i) We first assume that $g_t(0, 0, 0) = 0$.

Consider an arbitrary decomposition of ξ

$$\xi = \sum_{i=1}^n \xi_i \text{ such that } \|\xi_i\|_\infty \leq \frac{1}{2\sqrt{15}\sqrt{2670}\mu e^{\frac{3}{2}CT}}, \text{ for all } i.$$

We will now construct a solution to (2.2) recursively.

Step 1 We define $g^1 := g$ and (Y^1, Z^1, U^1) as the unique solution of

$$Y_t^1 = \xi_1 + \int_t^T g_s^1(Y_s^1, Z_s^1, U_s^1) ds - \int_t^T Z_s^1 dB_s - \int_t^T \int_E U_s^1(x) \tilde{\mu}(ds, dx), \quad \mathbb{P} - a.s. \quad (2.16)$$

Let us show why this solution exists. Since g^1 satisfies Assumption 2.4, we know by Lemma 2.3 that it satisfies Assumption 2.3 with $\phi_t := D_z g_t(y, 0, u)$ and $\psi_t := D_u g_t(y, z, 0)$, these processes being respectively in $\mathbb{H}_{\text{BMO}}^2$ and $\mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ by assumption. Furthermore, we have $\psi_t(x) \geq C_1(1 \wedge |x|)$ with $C_1 \geq -1 + \delta$. Thanks to Theorem 2.1 and with the notations of Lemma 2.2, we can then define the solution to the BSDEJ with driver \bar{g}^1 (which still satisfies $\bar{g}^1(0, 0, 0) = 0$) and terminal condition ξ_1 under the probability measure \mathbb{Q}^1 defined by

$$\frac{d\mathbb{Q}^1}{d\mathbb{P}} = \mathcal{E} \left(\int_0^T \phi_s dB_s + \int_0^T \int_E \psi_s(x) \tilde{\mu}(dx, ds) \right).$$

Thanks to Lemma 2.2, this gives us a solution (Y^1, Z^1, U^1) to (2.16) with Y^1 bounded, which in turn implies with Lemma 2.1 that $(Z^1, U^1) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$.

Step 2 We assume that we have constructed similarly $(Y^j, Z^j, U^j) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ for $j \leq i-1$. We then define the generator

$$g_t^i(y, z, u) := g_t \left(\bar{Y}_t^{i-1} + y, \bar{Z}_t^{i-1} + z, \bar{U}_t^{i-1} + u \right) - g_t \left(\bar{Y}_t^{i-1}, \bar{Z}_t^{i-1}, \bar{U}_t^{i-1} \right),$$

where

$$\bar{Y}_t^{i-1} := \sum_{j=1}^{i-1} Y_t^j, \quad \bar{Z}_t^{i-1} := \sum_{j=1}^{i-1} Z_t^j, \quad \bar{U}_t^{i-1} := \sum_{j=1}^{i-1} U_t^j.$$

Notice that $g^i(0, 0, 0) = 0$ and since g satisfies Assumption 2.1(iii), we have the estimate

$$\begin{aligned} g_t^i(y, z, u) &\leq 2\alpha_t + \beta \left| y + \bar{Y}_t^{i-1} \right| + \beta \left| \bar{Y}_t^{i-1} \right| + \gamma \left| z + \bar{Z}_t^{i-1} \right|^2 + \gamma \left| \bar{Z}_t^{i-1} \right|^2 \\ &\quad + \frac{1}{\gamma} j_t \left(\gamma \left(u + \bar{U}_t^{i-1} \right) \right) + \frac{1}{\gamma} j_t \left(\gamma \bar{U}_t^{i-1} \right) \\ &\leq 2\alpha_t + 2\beta \left| \bar{Y}_t^{i-1} \right| + 3\gamma \left| \bar{Z}_t^{i-1} \right|^2 + \frac{1}{\gamma} j_t \left(\gamma \bar{U}_t^{i-1} \right) + \frac{1}{2\gamma} j_t \left(2\gamma \bar{U}_t^{i-1} \right) \\ &\quad + \beta |y| + 2\gamma |z|^2 + \frac{1}{2\gamma} j_t (2\gamma u), \end{aligned}$$

where we used the inequalities $(a+b)^2 \leq 2(a^2 + b^2)$ and (4.10) below with $\gamma_1 = \gamma_2 := 1/(2\gamma)$.

Then, since $(\bar{Y}^{i-1}, \bar{Z}^{i-1}, \bar{U}^{i-1}) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$, we know that the term which does not depend on (y, z, u) above satisfies the same integrability condition as $g_t(0, 0, 0)$ in (2.6) (see also the arguments we used in Remark 2.4). Therefore, since $g^i(0, 0, 0) = 0$, we have one side of the

inequality in Assumption 2.1(iii), and the other one can be proved similarly. This yields that g^i satisfies Assumption 2.1.

Similarly as in Step 1, we will now show that there exists a solution $(Y^i, Z^i, U^i) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ to the BSDEJ

$$Y_t^i = \xi_i + \int_t^T g_s^i(Y_s^i, Z_s^i, U_s^i) ds - \int_t^T Z_s^i dB_s - \int_t^T \int_E U_s^i(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s. \quad (2.17)$$

Since g satisfies Assumptions 2.4, we can define

$$\phi_t^i := D_z g_t^i(y, 0, u) = D_z g_t(y, \bar{Z}_t^{i-1}, u), \quad \psi_t^i := D_u g_t^i(y, z, 0) = D_u g_t(y, z, \bar{U}_t^{i-1}).$$

We then know that

$$|\phi_t^i| \leq r_t + \theta |\bar{Z}_t^{i-1}|, \quad |\psi_t^i| \leq m_t + \theta |\bar{U}_t^{i-1}|, \quad \psi_t^i(x) \geq C_1(1 \wedge |x|) \geq -1 + \delta.$$

Since by hypothesis $(\bar{Z}^{i-1}, \bar{U}^{i-1}) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$, we can define a probability measure \mathbb{Q}^i by

$$\frac{d\mathbb{Q}^i}{d\mathbb{P}} = \mathcal{E} \left(\int_0^T \phi_s^i dB_s + \int_0^T \int_E \psi_s^i(x) \tilde{\mu}(dx, ds) \right).$$

Now, using the notations of Lemma 2.2, we define a generator \bar{g}^i from g^i (which still satisfies $\bar{g}^i(0, 0, 0) = 0$). It is then easy to check that \bar{g}^i satisfies Assumption 2.3. Therefore, by Theorem 2.1, we obtain the existence of a solution to the BSDE with generator \bar{g}^i and terminal condition ξ_i under \mathbb{Q}^i . Using Lemma 2.2, this provides a solution (Y^i, Z^i, U^i) with Y^i bounded to the BSDE (2.17). By Lemma 2.1 and since g^i satisfies Assumption 2.1, the boundedness of Y^i implies that $(Z^i, U^i) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ and therefore that $(\bar{Y}^i, \bar{Z}^i, \bar{U}^i) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$.

Step 3 Finally, by summing the BSDEJs (2.17), we obtain

$$\bar{Y}^n = \xi + \int_t^T g_s(\bar{Y}_s^n, \bar{Z}_s^n, \bar{U}_s^n) ds - \int_t^T \bar{Z}_s^n dB_s - \int_t^T \int_E \bar{U}_s^n(x) \tilde{\mu}(dx, ds).$$

Since \bar{Y}^n is bounded (because the Y^i are all bounded), Lemma 2.1 implies that $(\bar{Z}^n, \bar{U}^n) \in \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$, which ends the proof.

(ii) In the general case $g_t(0, 0, 0) \neq 0$, we can argue exactly as in Corollary 2.2 (see also Proposition 2 in [36]) to obtain the result. \square

2.8 A uniqueness result

We emphasize that the above theorems provide an existence result for every bounded terminal condition, but we only have uniqueness when the infinite norm of ξ is small enough. In order to have a general uniqueness result, we add the following assumptions, which were first introduced by Royer [34] and Briand and Hu [8]

Assumption 2.5. *For every (y, z, u, u') there exists a predictable and \mathcal{E} -mesurable process (γ_t) such that*

$$g_t(y, z, u) - g_t(y, z, u') \leq \int_E \gamma_t(x)(u - u')(x) \nu_t(dx),$$

where there exist constants $C_2 > 0$ and $C_1 \geq -1 + \delta$ for some $\delta > 0$ such that

$$C_1(1 \wedge |x|) \leq \gamma_t(x) \leq C_2(1 \wedge |x|).$$

Assumption 2.6. g is jointly convex in (z, u) .

We then have the following result

Theorem 2.3. Assume that $\xi \in \mathbb{L}^\infty$, and that the generator g satisfies either

- (i) Assumptions 2.1, 2.4(i),(ii) and 2.5.
- (ii) Assumptions 2.1, 2.4 and 2.6, and that $g(0,0,0)$ and the process α appearing in Assumption 2.1(iii) are bounded by some constant $M > 0$.

Then there exists a unique solution to the BSDEJ (2.2).

In order to prove this Theorem, we will use the following comparison Theorem for solutions of BSDEJs

Proposition 2.7. Let ξ^1 and ξ^2 be two \mathcal{F}_T -measurable random variables. Let g^1 be a function satisfying either

- (i) Assumptions 2.1, 2.3(i),(ii) and 2.5.
- (ii) Assumptions 2.1, 2.3(i) and 2.6, and that $|g^1(0,0,0)| + \alpha \leq M$ where α is the process appearing in Assumption 2.1(iii) and M is a positive constant.

Let g^2 be another function and for $i = 1, 2$, let (Y^i, Z^i, U^i) be the solution of the BSDEJ with terminal condition ξ^i and generator g^i (we assume that existence holds in our spaces), that is to say for every $t \in [0, T]$

$$Y_t^i = \xi^i + \int_t^T g_s^i(Y_s^i, Z_s^i, U_s^i) ds - \int_t^T Z_s^i dB_s - \int_t^T \int_E U_s^i(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s.$$

Assume further that $\xi^1 \leq \xi^2$, $\mathbb{P} - a.s.$ and $g_t^1(Y_t^2, Z_t^2, U_t^2) \leq g_t^2(Y_t^2, Z_t^2, U_t^2)$, $\mathbb{P} - a.s.$ Then $Y_t^1 \leq Y_t^2$, $\mathbb{P} - a.s.$

Moreover in case (i), if in addition we have $Y_0^1 = Y_0^2$, then for all t , $Y_t^1 = Y_t^2$, $Z_t^1 = Z_t^2$ and $U_t^1 = U_t^2$, $\mathbb{P} - a.s.$

Remark 2.9. Of course, we can replace the convexity property in Assumption 2.5 by concavity without changing the results of Proposition 2.7. Indeed, if Y is a solution to the BSDE with convex generator g and terminal condition ξ , then $-Y$ is a solution to the BSDE with concave generator $\tilde{g}(y, z, u) := -g(-y, -z, -u)$ and terminal condition $-\xi$. then we can apply the results of Proposition 2.7.

Proof.

Step 1 In order to prove (i), let us note

$$\begin{aligned} \delta Y &:= Y^1 - Y^2, \quad \delta Z := Z^1 - Z^2, \quad \delta U := U^1 - U^2, \quad \delta \xi := \xi^1 - \xi^2 \\ \delta g_t &:= g_t^1(Y_t^2, Z_t^2, U_t^2) - g_t^2(Y_t^2, Z_t^2, U_t^2). \end{aligned}$$

Using Assumption 2.3(i), (ii), we know that there exists a bounded process λ and a process η with

$$|\eta_s| \leq \mu(|Z_s^1| + |Z_s^2|), \quad (2.18)$$

such that

$$\begin{aligned}
\delta Y_t &= \delta \xi + \int_t^T \delta g_s ds + \int_t^T \lambda_s \delta Y_s ds + \int_t^T (\eta_s + \phi_s) \delta Z_s ds \\
&+ \int_t^T g_s^1(Y_s^1, Z_s^1, U_s^1) - g_s^1(Y_s^1, Z_s^1, U_s^2) ds - \int_t^T \int_E \delta U_s(x) \gamma_s(x) \nu_s(dx) ds \\
&+ \int_t^T \int_E \delta U_s(x) \gamma_s(x) \nu_s(dx) ds - \int_t^T \delta Z_s dB_s - \int_t^T \int_E \delta U_s(x) \tilde{\mu}(dx, ds), \tag{2.19}
\end{aligned}$$

where γ is the predictable process appearing in the right hand side of Assumption 2.5.

Define for $s \geq t$, $e^{\Lambda_s} := e^{\int_t^s \lambda_u du}$, and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E} \left(\int_t^s (\eta_s + \phi_s) dB_s + \int_t^s \int_E \gamma_s(x) \tilde{\mu}(dx, ds) \right).$$

Since the Z^i are in $\mathbb{H}_{\text{BMO}}^2$, so is η and by our assumption on γ_s the above stochastic exponential defines a true strictly positive uniformly integrable martingale (see Kazamaki [15]). Then applying Itô's formula and taking conditional expectation under the probability measure \mathbb{Q} , we obtain

$$\begin{aligned}
\delta Y_t &= \mathbb{E}_t^{\mathbb{Q}} \left[e^{\Lambda_T} \delta \xi + \int_t^T e^{\Lambda_s} \delta g_s ds \right] \\
&+ \mathbb{E}_t^{\mathbb{P}} \left[\int_t^T e^{\Lambda_s} \left(g_s^1(Y_s^1, Z_s^1, U_s^1) - g_s^1(Y_s^1, Z_s^1, U_s^2) - \int_E \gamma_s(x) \delta U_s(x) \nu_s(dx) \right) ds \right] \leq 0, \tag{2.20}
\end{aligned}$$

using Assumption 2.5.

Step 2 The proof of the comparison result when (ii) holds is a generalization of Theorem 5 in [8]. However, due to the presence of jumps our proof is slightly different. For the convenience of the reader, we will highlight the main differences during the proof.

For any $\theta \in (0, 1)$ let us denote

$$\delta Y_t := Y_t^1 - \theta Y_t^2, \quad \delta Z_t := Z_t^1 - \theta Z_t^2, \quad \delta U_t := U_t^1 - \theta U_t^2, \quad \delta \xi := \xi^1 - \theta \xi^2.$$

First of all, we have for all $t \in [0, T]$

$$\delta Y_t = \delta \xi + \int_t^T G_s ds - \int_t^T \delta Z_s dB_s - \int_t^T \int_E \delta U_s(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s.,$$

where

$$G_t := g_t^1(Y_t^1, Z_t^1, U_t^1) - \theta g_t^2(Y_t^2, Z_t^2, U_t^2).$$

We emphasize that unlike in [8], we have not linearized the generator in y using the Assumption 2.3(i). It will be clear later on why.

We will now bound G_t from above. First, we rewrite it as

$$G_t = G_t^1 + G_t^2 + G_t^3,$$

where

$$\begin{aligned}
G_t^1 &:= g_t^1(Y_t^1, Z_t^1, U_t^1) - g_t^1(Y_t^2, Z_t^1, U_t^1), \quad G_t^2 := g_t^1(Y_t^2, Z_t^1, U_t^1) - \theta g_t^1(Y_t^2, Z_t^2, U_t^2) \\
G_t^3 &:= \theta (g_t^1(Y_t^2, Z_t^2, U_t^2) - g_t^2(Y_t^2, Z_t^2, U_t^2)).
\end{aligned}$$

Then, we have using Assumption 2.3(i)

$$\begin{aligned} G_t^1 &= g_t^1(Y_t^1, Z_t^1, U_t^1) - g_t^1(\theta Y_t^2, Z_t^1, U_t^1) + g_t^1(\theta Y_t^2, Z_t^1, U_t^1) - g_t^1(Y_t^2, Z_t^1, U_t^1) \\ &\leq C(|\delta y_t| + (1 - \theta)|y_t^2|). \end{aligned} \quad (2.21)$$

Next, we estimate G^2 using Assumption 2.1 and the convexity in (z, u) of g^1

$$\begin{aligned} g_t^1(Y_t^2, Z_t^1, U_t^1) &= g_t^1\left(Y_t^2, \theta Z_t^2 + (1 - \theta)\frac{Z_t^1 - \theta Z_t^2}{1 - \theta}, \theta U_t^2 + (1 - \theta)\frac{U_t^1 - \theta U_t^2}{1 - \theta}\right) \\ &\leq \theta g_t^1(Y_t^2, Z_t^2, U_t^2) + (1 - \theta)g_t^1\left(Y_t^2, \frac{\delta Z_t}{1 - \theta}, \frac{\delta U_t}{1 - \theta}\right) \\ &\leq \theta g_t^1(Y_t^2, Z_t^2, U_t^2) + (1 - \theta)(M + \beta|Y_t^2|) + \frac{\gamma}{2(1 - \theta)}|\delta Z_t|^2 + \frac{1 - \theta}{\gamma}j_t\left(\frac{\gamma}{1 - \theta}\delta U_t\right). \end{aligned}$$

Hence

$$G_t^2 \leq (1 - \theta)(M + \beta|Y_t^2|) + \frac{\gamma}{2(1 - \theta)}|\delta Z_t|^2 + \frac{1 - \theta}{\gamma}j_t\left(\frac{\gamma}{1 - \theta}\delta U_t\right). \quad (2.22)$$

Finally, G^3 is negative by assumption. Therefore, using (2.21) and (2.22), we obtain

$$G_t \leq C|\delta Y_t| + (1 - \theta)(M + (\beta + C)|Y_t^2|) + \frac{\gamma}{2(1 - \theta)}|Z_t|^2 + \frac{1 - \theta}{\gamma}j_t\left(\frac{\gamma}{1 - \theta}\delta U_t\right). \quad (2.23)$$

Now we will get rid off the quadratic and exponential terms in z and u using a classical exponential change. Let us then denote for some $\nu > 0$

$$P_t := e^{\nu \delta Y_t}, \quad Q_t := \nu e^{\nu \delta Y_t} \delta Z_t, \quad R_t(x) := e^{\nu \delta Y_t} \left(e^{\nu \delta U_t(x)} - 1 \right).$$

By Itô's formula we obtain for every $t \in [0, T]$, $\mathbb{P} - a.s.$

$$P_t = P_T + \int_t^T \nu P_s \left(G_s - \frac{\nu}{2} |\delta Z_s|^2 - \frac{1}{\nu} j_s(\nu \delta U_s) \right) ds - \int_t^T Q_s dB_s - \int_t^T \int_E R_s(x) \tilde{\mu}(dx, ds).$$

Now choose $\nu = \gamma/(1 - \theta)$. We emphasize that this is here that the presence of jumps forces us to change our proof in comparison with the one in [8]. Indeed, if we had immediately linearized in y then we could not have chosen ν constant such that the quadratic and exponentials terms in (2.23) would disappear. This is not a problem in [8], since they can choose ν of the form $M/(1 - \theta)$ with M large enough and still make the quadratic term in z disappear. However, in the jump case, the application $\gamma \mapsto \gamma^{-1}j_t(\gamma u)$ is not always increasing, and this trick does not work. Nonetheless, we now define the strictly positive and continuous process

$$D_t := \exp \left(\gamma \int_0^t \left(M + (\beta + C)|Y_s^2| + \frac{C}{1 - \theta} |\delta Y_s| \right) ds \right).$$

Applying Itô's formula to $D_t P_t$, we obtain

$$\begin{aligned} d(D_s P_s) &= -\nu D_s P_s \left(G_s - \frac{\nu}{2} |\delta Z_s|^2 - \frac{1}{\nu} j_s(\nu \delta U_s) - C |\delta Y_s| + (1 - \theta)(M + (\beta + C)|Y_s^2|) \right) ds \\ &\quad + D_s Q_s dB_s + \int_E D_s R_s(x) \tilde{\mu}(dx, ds). \end{aligned}$$

Hence, using the inequality (2.23), we deduce

$$D_t P_t \leq \mathbb{E}_t^{\mathbb{P}} [D_T P_T], \quad \mathbb{P} - a.s.,$$

which can be rewritten

$$\delta Y_t \leq \frac{1-\theta}{\gamma} \ln \left(\mathbb{E}_t^{\mathbb{P}} \left[\exp \left(\gamma \int_t^T \left(M + (\beta + C) |Y_s^2| + \frac{C}{1-\theta} |\delta Y_s| \right) ds + \frac{\gamma}{1-\theta} \delta \xi \right) \right] \right), \mathbb{P} - a.s.$$

Next, we have

$$\delta \xi = (1-\theta)\xi^1 + \theta(\xi^1 - \xi^2) \leq (1-\theta)|\xi^1|.$$

Consequently, we have for some constant $C_0 > 0$, independent of θ , using the fact that Y^2 and ξ^1 are bounded $\mathbb{P} - a.s.$

$$\delta Y_t \leq \frac{1-\theta}{\gamma} \left(\ln(C_0) + \ln \left(\mathbb{E}_t^{\mathbb{P}} \left[\exp \left(\frac{C}{1-\theta} \int_t^T |\delta Y_s| ds \right) \right] \right) \right), \mathbb{P} - a.s. \quad (2.24)$$

We finally argue by contradiction. More precisely, let

$$\mathcal{A} := \{\omega \in \Omega, Y_t^1(\omega) > Y_t^2(\omega)\},$$

and assume that $\mathbb{P}(\mathcal{A}) > 0$. Let us then call \mathcal{N} the \mathbb{P} -negligible set outside of which (2.24) holds. Since \mathcal{A} has a strictly positive probability, $\mathcal{B} := \mathcal{A} \cap (\Omega \setminus \mathcal{N})$ is not empty and also has a strictly positive probability. Then, we would have from (2.24) that for every $\omega \in \mathcal{B}$

$$\delta Y_t(\omega) \leq \frac{1-\theta}{\gamma} \ln(C_0) + \frac{C}{\gamma} \int_t^T \|\delta Y_s\|_{\infty, \mathcal{B}} ds, \quad (2.25)$$

where $\|\cdot\|_{\infty, \mathcal{B}}$ is the usual infinite norm restricted to \mathcal{B} .

Now, using the dominated convergence theorem, we can let $\theta \uparrow 1^-$ in (2.25) to obtain that for any $\omega \in \mathcal{B}$

$$Y_t^1(\omega) - Y_t^2(\omega) \leq \frac{C}{\gamma} \int_t^T \|Y_s^1 - Y_s^2\|_{\infty, \mathcal{B}} ds,$$

which in turns implies, since $\mathcal{B} \subset \mathcal{A}$

$$\|Y_t^1 - Y_t^2\|_{\infty, \mathcal{B}} \leq \frac{C}{\gamma} \int_t^T \|Y_s^1 - Y_s^2\|_{\infty, \mathcal{B}} ds.$$

But with Gronwall's lemma this implies that $\|Y_t^1 - Y_t^2\|_{\infty, \mathcal{B}} = 0$ and the desired contradiction. Hence the result.

Step 3 Let us now assume that $Y_0^1 = Y_0^2$ and that we are in the same framework as in Step 1. Using this in (2.20) above when $t = 0$, we obtain

$$\begin{aligned} 0 &= \mathbb{E}^{\mathbb{Q}} \left[e^{\Lambda_T} \delta \xi + \int_0^T e^{\Lambda_s} \delta g_s ds \right] \\ &+ \mathbb{E}^{\mathbb{Q}} \left[\int_0^T e^{\Lambda_s} \left(g_s^1(Y_s^1, Z_s^1, U_s^1) - g_s^1(Y_s^1, Z_s^1, U_s^2) - \int_E \delta U_s(x) \gamma_s(x) \nu_s(dx) \right) ds \right] \leq 0. \end{aligned} \quad (2.26)$$

Hence, since all the above quantities have the same sign, this implies in particular that

$$e^{\Lambda_T} \delta \xi + \int_0^T e^{\Lambda_s} \delta g_s ds = 0, \mathbb{P} - a.s.$$

Moreover, we also have $\mathbb{P} - a.s.$

$$\int_0^T e^{\Lambda_s} (g_s^1(Y_s^1, Z_s^1, U_s^1) - g_s^1(Y_s^1, Z_s^1, U_s^2)) ds = \int_0^T e^{\Lambda_s} \left(\int_E \delta U_s(x) \gamma_s(x) \nu_s(dx) \right) ds.$$

Using this result in (2.19), we obtain with Itô's formula

$$\begin{aligned} \delta Y_t = & \int_0^T e^{\Lambda_s} \left(\int_E \delta U_s(x) \gamma_s(x) \nu_s(dx) \right) ds - \int_t^T e^{\Lambda_s} \delta Z_s (dB_s - (\eta_s + \phi_s) ds) \\ & - \int_t^T \int_E e^{\Lambda_s} \delta U_s(x) \tilde{\mu}(dx, ds). \end{aligned} \quad (2.27)$$

The right-hand side is a martingale under \mathbb{Q} with null expectation. Thus, since $\delta Y_t \leq 0$, this implies that $Y_t^1 = Y_t^2$, $\mathbb{P} - a.s.$ Using this in (2.27), we obtain that the martingale part must be equal to 0, which implies that $\delta Z_t = 0$ and $\delta U_t = 0$. \square

Remark 2.10. *In the above proof of the comparison theorem in case (i), we emphasize that it is actually sufficient that, instead of Assumption 2.5, the generator g satisfies*

$$g_s^1(Y_s^1, Z_s^1, U_s^1) - g_s^1(Y_s^1, Z_s^1, U_s^2) \leq \int_E \gamma_s(x) \delta U_s(x) \nu_s(dx),$$

for some γ_s such that

$$C_1(1 \wedge |x|) \leq \gamma_s(x) \leq C_2(1 \wedge |x|).$$

Besides, this also holds true for the comparison Theorem for Lipschitz BSDEs with jumps proved by Royer (see Theorem 2.5 in [34]).

We can now prove Theorem 2.3

Proof. [Proof of Theorem 2.3] First let us deal with the question of existence.

- (i) If g satisfies Assumptions 2.1, 2.4(i),(ii) and 2.5, the existence part can be obtained exactly as in the previous proof, starting from a small terminal condition, and using the fact that Assumption 2.5 implies that g is Lipschitz in u . Thus we omit it.
- (ii) If g satisfies Assumptions 2.1 and 2.4, then we already proved existence for bounded terminal conditions.

The uniqueness is then a simple consequence of the above comparison theorem. \square

Remark 2.11. *As a consequence of the nonlinear Doob-Meyer decomposition that we prove in the next Section, we will obtain a reverse comparison Theorem in Corollary 3.1.*

2.9 A priori estimates and stability

In this subsection, we show that under our hypotheses, we can obtain *a priori* estimates for quadratic BSDEs with jumps. We have the following results

Proposition 2.8. *Let $(\xi^1, \xi^2) \in \mathbb{L}^\infty \times \mathbb{L}^\infty$ and let g be a function satisfying Assumptions 2.1, 2.3(i),(ii) and 2.5. Let us consider for $i = 1, 2$ the solutions $(Y^i, Z^i, U^i) \in \mathcal{S}^\infty \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2$ of the BSDEs with generator g and terminal condition ξ^i (once again existence is assumed). Then we have for some constant $C > 0$*

$$\begin{aligned} \|Y^1 - Y^2\|_{\mathcal{S}^\infty} + \|U^1 - U^2\|_{L^\infty(\nu)} &\leq C \|\xi^1 - \xi^2\|_\infty \\ \|Z^1 - Z^2\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|U^1 - U^2\|_{\mathbb{J}_{\text{BMO}}^2}^2 &\leq C \|\xi^1 - \xi^2\|_\infty. \end{aligned}$$

Proof. Following exactly the same arguments as in Step 1 of the proof Proposition 2.7, we obtain with the same notations

$$Y_t^1 - Y_t^2 = \mathbb{E}_t^{\mathbb{Q}} [e^{\Lambda T} (\xi^1 - \xi^2)] \leq C \|\xi^1 - \xi^2\|_{\infty}, \quad \mathbb{P} - a.s.$$

Notice then that this implies as usual that there is a version of $(U^1 - U^2)$ (still denoted $(U^1 - U^2)$ for simplicity) which is bounded by $2 \|Y^1 - Y^2\|_{\mathcal{S}_{\infty}}$. This gives easily the first estimate.

Let now $\tau \in \mathcal{T}_0^T$ be a stopping time. Denote also

$$\delta g_s := g_s(Y_s^1, Z_s^1, U_s^1) - g_s(Y_s^2, Z_s^2, U_s^2).$$

By Itô's formula, we have using standard calculations

$$\begin{aligned} \mathbb{E}_{\tau}^{\mathbb{P}} \left[\int_{\tau}^T |Z_s|^2 ds + \int_{\tau}^T \int_E U_s^2(x) \nu_s(dx) ds \right] &\leq \mathbb{E}_{\tau}^{\mathbb{P}} \left[|\xi^1 - \xi^2|^2 + 2 \int_{\tau}^T (Y_s^1 - Y_s^2) \delta g_s ds \right] \\ &\leq \|\xi^1 - \xi^2\|_{\infty}^2 + 2 \|Y^1 - Y^2\|_{\mathcal{S}_{\infty}} \mathbb{E}_{\tau}^{\mathbb{P}} \left[\int_{\tau}^T |\delta g_s| ds \right]. \end{aligned} \quad (2.28)$$

Then, using Assumption 2.1, we estimate

$$\begin{aligned} |\delta g_t| &\leq C \left(|g_t(0, 0, 0)| + \alpha_t + \sum_{i=1,2} |Y_t^i| + |Z_t^i|^2 + j_t(\gamma U^i) + j_t(-\gamma U^i) \right) \\ &\leq C \left(|g_t(0, 0, 0)| + \alpha_t + \sum_{i=1,2} |Y_t^i| + |Z_t^i|^2 + \|U_t^i\|_{L^2(\nu)}^2 \right), \end{aligned}$$

where we used the fact that for every x in a compact subset of \mathbb{R} , $0 \leq e^x - 1 - x \leq Cx^2$. Using this estimate and the integrability assumed on $g_t(0, 0, 0)$ and α_t in (2.28) entails

$$\begin{aligned} &\mathbb{E}_{\tau}^{\mathbb{P}} \left[\int_{\tau}^T |Z_s|^2 ds + \int_{\tau}^T \int_E U_s^2(x) \nu_s(dx) ds \right] \\ &\leq \|\xi^1 - \xi^2\|_{\infty}^2 + C \|\xi^1 - \xi^2\|_{\infty} \left(1 + \sum_{i=1,2} \|Y^i\|_{\mathcal{S}_{\infty}} + \|Z^i\|_{\mathbb{H}_{\text{BMO}}^2} + \|U^i\|_{\mathbb{J}_{\text{BMO}}^2} \right) \\ &\leq C \|\xi^1 - \xi^2\|_{\infty}, \end{aligned}$$

which ends the proof. \square

Proposition 2.9. *Let $(\xi^1, \xi^2) \in \mathbb{L}^{\infty} \times \mathbb{L}^{\infty}$ and let g be a function satisfying Assumptions 2.1, 2.3(i) and 2.6 and such that $|g(0, 0, 0)| + \alpha \leq M$ where α is the process appearing in Assumption 2.1(iii) and M is a positive constant. Let us consider for $i = 1, 2$ the solutions $(Y^i, Z^i, U^i) \in \mathcal{S}^{\infty} \times \mathbb{H}_{\text{BMO}}^2 \times \mathbb{J}_{\text{BMO}}^2$ of the BSDEs with generator g and terminal condition ξ^i (once again existence is assumed). Then we have for some constant $C > 0$*

$$\begin{aligned} \|Y^1 - Y^2\|_{\mathcal{S}_{\infty}} + \|U^1 - U^2\|_{L^{\infty}(\nu)} &\leq C \|\xi^1 - \xi^2\|_{\infty} \\ \|Z^1 - Z^2\|_{\mathbb{H}_{\text{BMO}}^2}^2 + \|U^1 - U^2\|_{\mathbb{J}_{\text{BMO}}^2}^2 &\leq C \|\xi^1 - \xi^2\|_{\infty}. \end{aligned}$$

Proof. Following Step 2 of the proof of Proposition 2.7, we obtain for any $\theta \in (0, 1)$

$$\frac{Y_t^1 - \theta Y_t^2}{1 - \theta} \leq \frac{1}{\gamma} \ln \left(\mathbb{E}_t^{\mathbb{P}} \left[\exp \left(\gamma \int_t^T \left(M + (\beta + C) |Y_s^2| + \frac{C |Y_s^1 - \theta Y_s^2|}{1 - \theta} \right) ds + \frac{\gamma(\xi^1 - \theta \xi^2)}{1 - \theta} \right) \right] \right),$$

and of course by symmetry, the same holds if we interchange the roles of the exponents 1 and 2.

Since all the quantites above are bounded, we obtain easily after some calculations and after letting $\theta \uparrow 1^-$

$$Y_t^1 - Y_t^2 \leq C \left(\|\xi^1 - \xi^2\|_{\infty} + \int_t^T \|Y_s^1 - Y_s^2\|_{\infty} ds \right), \quad \mathbb{P} - a.s.,$$

and symmetrically

$$Y_t^2 - Y_t^1 \leq C \left(\|\xi^1 - \xi^2\|_{\infty} + \int_t^T \|Y_s^1 - Y_s^2\|_{\infty} ds \right), \quad \mathbb{P} - a.s.$$

Hence, we can use Gronwall's lemma to obtain

$$\|Y^1 - Y^2\|_{\mathcal{S}_{\infty}} \leq C \|\xi^1 - \xi^2\|_{\infty}.$$

All the other estimates can then be obtained as in the proof of Proposition 2.8. \square

3 Quadratic g -martingales with jumps

The theory of g -expectations was introduced by Peng in [31] as an example of non-linear expectations. Since then, numerous authors have generalized his results, extending them notably to the case of quadratic coefficients (see Ma and Yao [26]). An extension to discontinuous filtrations was obtained by Royer [34] and Lin [25]. In particular, Royer [34] gave domination conditions under which we can write a non-linear expectation as a g -expectation. We refer the interested reader to these papers for more details about these filtration-consistent operators, and we recall for simplicity some of their general properties below.

Let us start with a general definition.

Definition 3.1. Let $\xi \in \mathbb{L}^{\infty}$ and let g be such that the BSDEJ with generator g and terminal condition ξ has a unique solution and such that comparison in the sense of Proposition 2.7 holds (for instance g could satisfy any of the conditions in Theorem 2.3). Then for every $t \in [0, T]$, we define the conditional g -expectation of ξ as follows

$$\mathcal{E}_t^g[\xi] := Y_t,$$

where (Y, Z, U) solves the following BSDEJ

$$Y_t = \xi + \int_t^T g_t(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds).$$

Remark 3.1. Notice that $\mathcal{E}^g : \mathbb{L}^{\infty}(\Omega, \mathcal{F}_T, \mathbb{P}) \rightarrow \mathbb{L}^{\infty}(\Omega, \mathcal{F}_t, \mathbb{P})$ does not define a true operator. Indeed, to each bounded \mathcal{F}_T -measurable random variable ξ , we associate the value Y_t , which is defined \mathbb{P} -a.s., that is to say outside a \mathbb{P} -negligible set N , but this set N depends on ξ . We cannot a priori find a common negligible set for all variables in \mathbb{L}^{∞} , and then define an operator \mathcal{E}^g on a fixed domain, except if we only consider a countable set of variables ξ on which acts \mathcal{E}^g .

We have a notion of g -martingales and g -sub(super)martingales.

Definition 3.2. $X \in \mathcal{S}^\infty$ is called a g -submartingale (resp. g -supermartingale) if

$$\mathcal{E}_s^g[X_t] \geq (\text{resp. } \leq) X_s, \quad \mathbb{P} - \text{a.s.}, \quad \text{for any } 0 \leq s \leq t \leq T.$$

X is called a g -martingale if it is both a g -sub and supermartingale.

The following results are easy generalizations of the classical arguments which can be found in [31] or [3], and are consequences of the comparison theorem. We therefore omit the proofs.

Lemma 3.1. $\{\mathcal{E}_t^g\}_{t \geq 0}$ is monotonic increasing and time consistent, i.e.

- $\xi_1 \geq \xi_2, \mathbb{P}\text{-a.s.}$ implies that $\mathcal{E}_t^g(\xi_1) \geq \mathcal{E}_t^g(\xi_2), \mathbb{P}\text{-a.s.}, \forall t \geq 0$.
- For any bounded stopping times $R \leq S \leq \tau$ and \mathcal{F}_τ -measurable random variable ξ_τ ,

$$\mathcal{E}_R^g(\mathcal{E}_S^g(\xi_\tau)) = \mathcal{E}_R^g(\xi_\tau) \quad \mathbb{P}\text{-a.s.} \quad (3.1)$$

Definition 3.3. We will say that \mathcal{E}^g is

- (i) Constant additive, if for any stopping times $R \leq S$, any \mathcal{F}_R -measurable random variable η_R and any \mathcal{F}_S -measurable random variable ξ_S ,

$$\mathcal{E}_R^g(\xi_S + \eta_R) = \mathcal{E}_R^g(\xi_S) + \eta_R, \quad \mathbb{P}\text{-a.s.}$$

- (ii) Positively homogeneous, if for any stopping times $R \leq S$, and any positive \mathcal{F}_R -measurable random variable λ ,

$$\mathcal{E}_R^g(\lambda \xi_S) = \lambda \mathcal{E}_R^g(\xi_S).$$

- (iii) Convex, if for any stopping times $R \leq S$, any random variables (ξ_S^1, ξ_S^2) and any $\lambda \in [0, 1]$,

$$\mathcal{E}_R^g(\lambda \xi_S^1 + (1 - \lambda) \xi_S^2) \leq \lambda \mathcal{E}_R^g(\xi_S^1) + (1 - \lambda) \mathcal{E}_R^g(\xi_S^2).$$

The next Lemma shows that the operator \mathcal{E}^g inherits the above properties from the generator g .

Lemma 3.2. (i) If g does not depend on y , then \mathcal{E}^g is constant additive.

- (ii) If g is positively homogeneous in (y, z, u) , then \mathcal{E}^g is positively homogeneous.

- (iii) If g is moreover right continuous on $[0, T)$ and continuous at T , then the reverse implications of (i) and (ii) are also true.

- (iv) \mathcal{E}^g is convex if g is convex in (y, z, u) .

- (v) If $g^1 \leq g^2, \mathbb{P}\text{-a.s.}$, then $\mathcal{E}^{g^1} \leq \mathcal{E}^{g^2}$. If g^1 and g^2 are moreover right continuous on $[0, T)$ and continuous at T , then the reverse is also true.

Proof.

We adapt the ideas of the proofs in [3] to our context with jumps.

(i) The proof of the first property is exactly the same as the proof of Theorem 6.7.b2 in [3], so we omit it.

(ii) Let $g^\lambda(t, y, z, u) := \frac{1}{\lambda}g_t(\lambda y, \lambda z, \lambda u)$. Then $\{\frac{1}{\lambda}\mathcal{E}_t^g(\lambda\xi_S)\}_{t \geq 0}$ is a solution of the BSDEJ with coefficient g^λ and terminal condition ξ_S . If $g = g^\lambda$ then

$$\frac{1}{\lambda}\mathcal{E}_t^g(\lambda\xi_S) = \mathcal{E}_t^g(\xi_S),$$

which is the desired result.

(iii) The reverse implications in (i) and (ii) are direct consequences of Corollary 3.1.

(iv) Suppose that g is convex in (y, z, u) . Let (Y^i, Z^i, U^i) be the unique solution of the BSDE with coefficients (g, ξ_S^i) , $i = 1, 2$, and set

$$\tilde{Y}_t = \lambda Y_t^1 + (1 - \lambda)Y_t^2, \quad \tilde{Z}_t = \lambda Z_t^1 + (1 - \lambda)Z_t^2 \quad \text{and} \quad \tilde{U}_t(\cdot) = \lambda U_t^1(\cdot) + (1 - \lambda)U_t^2(\cdot).$$

We have

$$\begin{aligned} -d\tilde{Y}_t &= [\lambda g_t(Y_t^1, Z_t^1, U_t^1) + (1 - \lambda)g_t(Y_t^2, Z_t^2, U_t^2)] dt - (\lambda Z_t^1 + (1 - \lambda)Z_t^2) dB_t \\ &\quad - \int_E (\lambda U_t^1(x) + (1 - \lambda)U_t^2(x)) \tilde{\mu}(dt, dx) \\ &= \left[g_t(\tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t) + k(t, Y_t^1, Y_t^2, Z_t^1, Z_t^2, U_t^1, U_t^2, \lambda) \right] dt - \tilde{Z}_t dB_t - \int_E \tilde{U}_t(x) \tilde{\mu}(dt, dx), \end{aligned}$$

where

$$k(t, Y_t^1, Y_t^2, Z_t^1, Z_t^2, U_t^1, U_t^2, \lambda) := \lambda g_t(Y_t^1, Z_t^1, U_t^1) + (1 - \lambda)g_t(Y_t^2, Z_t^2, U_t^2) - g_t(\tilde{Y}_t, \tilde{Z}_t, \tilde{U}_t),$$

is a non negative function. Then using Proposition 2.7 we obtain in particular

$$\mathcal{E}_t^g(\lambda\xi_S^1 + (1 - \lambda)\xi_S^2) \leq \tilde{Y}_t = \lambda\mathcal{E}_t^g(\xi_S^1) + (1 - \lambda)\mathcal{E}_t^g(\xi_S^2).$$

(v) This last property is a direct consequence of the comparison Theorem 2.7. The reverse implication is again a consequence of Corollary 3.1. \square

Example 3.1. *These easy properties allow us to construct examples of time consistent dynamic convex risk measures, by appropriate choices of generator g .*

- *As already pointed out in Remark 2.4, defining $g_t(z, u) := \frac{\gamma}{2}|z_t|^2 + \frac{1}{\gamma}j_t(\gamma u_t)$, we obtain the so called entropic risk measure on our particular filtration.*
- *As proved in [34], if we define*

$$g_t(z, u) := \eta|z| + \eta \int_E (1 \wedge |x|)u^+(x)\nu_t(dx) - C_1 \int_E (1 \wedge |x|)u^-(x)\nu_t(dx),$$

where $\eta > 0$ and $-1 < C_1 \leq 0$, then \mathcal{E}^g is a convex risk measure with the following representation $\mathcal{E}_0^g(\xi) = \sup_{\mathbb{Q} \in \mathbf{Q}} \mathbb{E}^{\mathbb{Q}}[\xi]$, with

$$\begin{aligned} \mathbf{Q} &:= \left\{ \mathbb{Q}, \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \mathcal{E} \left(\int_0^t \mu_s dB_s + \int_0^t \int_E v_s(x) \tilde{\mu}(ds, dx) \right) \right. \\ &\quad \left. \text{with } \mu \text{ and } v \text{ predictable, } |\mu_s| \leq \eta, v_s^+(x) \leq \eta(1 \wedge x), v_s^-(x) \leq C_1(1 \wedge x) \right\}. \end{aligned}$$

- If we define a linear generator g by

$$g_t(z, u) := \alpha z + \beta \int_E (1 \wedge |x|) u(x) \nu_t(dx), \quad \alpha \in \mathbb{R}, \beta \geq -1 + \delta \text{ for some } \delta > 0,$$

then we obtain a linear risk measure, since \mathcal{E}^g will only consist of a linear expectation with respect to the probability measure \mathbb{Q} , whose Radon-Nikodym derivative is equal to $\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(\alpha B_t + \int_0^t \int_E \beta(1 \wedge |x|) \tilde{\mu}(ds, dx)\right)$.

In the rest of this section, we will provide important properties of quadratic g -expectations and the associated g -martingales in discontinuous filtrations, which generalize the known results in simpler cases.

3.1 Non-linear Doob Meyer decomposition

We start by proving that the non-linear Doob Meyer decomposition first proved by Peng in [32] still holds in our context. We have two different sets of assumptions under which this result holds, and they are both related to the assumptions under which our comparison theorem 2.7 holds. From a technical point of view, our proof consists in approximating our generator by a sequence of Lipschitz generators. However, the novelty here is that because of the dependence of the generator in u , we cannot use the classical exponential transformation and then use some truncation arguments, as in [19] and [26]. Indeed, since u lives in an infinite dimensional space, those truncation type arguments no longer work *a priori*. Instead, inspired by [2], we will only use regularizations by inf-convolution, which are known to work in any Banach space.

Theorem 3.1. *Let Y be a càdlàg g -submartingale (resp. g -supermartingale) in \mathcal{S}^∞ (we assume that existence and uniqueness for the BSDEs with generator g hold for any bounded terminal condition). Assume further either one of these conditions*

- (i) *Assumptions 2.1 and 2.5 hold, with the addition that the process γ does not depend on (y, z) and that $|g(0, 0, 0)| + \alpha \leq M$, where α is the process appearing in Assumption 2.1(iii) and $M > 0$ is constant.*
- (ii) *Assumptions 2.1, 2.3(i) hold, g is concave (resp. convex) in (z, u) , $|g(0, 0, 0)| + \alpha \leq M$, where α is the process appearing in Assumption 2.1(iii) and $M > 0$ is constant.*

Then there exists a predictable non-decreasing (resp. non-increasing) process A null at 0 and processes $(Z, U) \in \mathbb{H}^2 \times \mathbb{J}^2$ such that

$$Y_t = Y_T + \int_t^T g_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds) - A_T + A_t, \quad t \in [0, T].$$

Remark 3.2. *We emphasize that the two assumptions in the above theorem are not of the same type. Indeed, Assumption 2.5 implies that the generator g is uniformly Lipschitz in u , which is a bit disappointing if we want to work in a quadratic context. This is why we also considered the convexity hypothesis on g , which allows us to retrieve a generator which is quadratic in both (z, u) . We do not know whether those two assumptions are necessary or not to obtain the result, but we remind the reader that our theorem encompasses the case of the so-called entropic generator, which has quadratic growth and is convex in (z, u) . To the best of our knowledge, this particular case which was already proved in [29], was the only result available in the literature up until now.*

Proof.

First of all, if Y is g -supermartingale, then $-Y$ is a g^- -submartingale where

$$g_t^-(y, z, u) := -g_t(-y, -z, -u).$$

Since g^- satisfies exactly the same Assumptions as g , and given that g^- is convex when g is concave, it is clear that we can without loss of generality restrict ourselves to the case of g -submartingales. We start with the first result.

Step 1: Assumptions 2.1 and 2.5 hold.

We will approximate the generator g by a sequence of functions (g^n) which are uniformly Lipschitz in (y, z) (recall that under the assumed assumptions, g is already Lipschitz in u). We emphasize that unlike most of the litterature on quadratic BSDEs, with the notable exception of [2] and [4], we will not use any exponential change in our proof.

Building upon the results of Lepeltier and San Martin [21], we would like to use a sup-convolution to regularize our generator. However, due to the quadratic growth assumption in z , such a sup-convolution is not always well defined. Therefore, we will first use a truncation argument to bound our generator from above by a function with linear growth. Let us thus define for all $n \geq 0$

$$\tilde{g}_t^n(y, z, u) := g_t(y, z, u) \wedge \left(M + n|z| - \frac{\gamma}{2}|z|^2 \right),$$

where the constants (α, γ) are the ones appearing in Assumption 2.4(ii).

It is clear that we have the following estimates

$$-M - \beta|y| - \frac{\gamma}{2}|z|^2 - \frac{1}{\gamma}j_t(-\gamma u) \leq \tilde{g}_t^n(y, z, u) \leq M + \beta|y| + n|z| + \frac{1}{\gamma}j_t(\gamma u),$$

and that \tilde{g}^n decreases pointwise to g .

We now define for all $p \geq n \vee \beta$

$$\tilde{g}_t^{n,p}(y, z, u) := \sup_{(w,v) \in \mathbb{Q}^{d+1}} \{ \tilde{g}_t^n(w, v, u) - p|y - w| - p|z - v| \}.$$

This function is indeed well-defined, since we have for $p \geq n$

$$\begin{aligned} \tilde{g}_t^{n,p}(y, z, u) &\leq M + \frac{1}{\gamma}j_t(\gamma u) + \sup_{(w,v) \in \mathbb{Q}^{d+1}} \{ \beta|w| + n|v| - p|y - w| - p|z - v| \} \\ &= M + \beta|y| + n|z| + \frac{1}{\gamma}j_t(\gamma u). \end{aligned}$$

Moreover, by the results of Lepeltier and San Martin [21], we know that $\tilde{g}^{n,p}$ is uniformly Lipschitz in (y, z) and that $\tilde{g}^{n,p}(y, z, u) \downarrow g_t(y, z, u)$ as n and p go to $+\infty$. Finally, we define

$$g_t^n(y, z, u) := \tilde{g}_t^{n,n}(y, z, u).$$

Then the g^n are uniformly Lipschitz in (y, z, u) and decrease pointwise to g . Now, we want somehow to use the fact that we know that the non-linear Doob-Meyer decomposition holds when the underlying generator is Lipschitz. But this was shown by Royer only when the generator also satisfies Assumption 2.5. Therefore, we will now verify that g^n inherits Assumption 2.5 from g . First of all, we show that this is true for \tilde{g}^n .

Let $u^1, u^2 \in L^\infty(\nu) \cap L^2(\nu)$ and fixe some $(y, z) \in \mathbb{R}^{d+1}$. Then if we have

$$g_t(y, z, u^1) \leq M + n|z| - \frac{\gamma}{2}|z|^2 \text{ and } g_t(y, z, u^2) \leq M + n|z| - \frac{\gamma}{2}|z|^2,$$

then

$$\tilde{g}_t^n(y, z, u^1) - \tilde{g}_t^n(y, z, u^2) = g_t(y, z, u^1) - g_t(y, z, u^2),$$

and the result is clear with the same process γ as the one for g .

Similarly, if

$$g_t(y, z, u^1) \geq M + n|z| - \frac{\gamma}{2}|z|^2 \text{ and } g_t(y, z, u^2) \geq M + n|z| - \frac{\gamma}{2}|z|^2,$$

then

$$\tilde{g}_t^n(y, z, u^1) - \tilde{g}_t^n(y, z, u^2) = 0,$$

and the desired result also follows by choosing the process γ in Assumption 2.5 to be 0.

Finally, if (the remaining case can be treated similarly)

$$g_t(y, z, u^1) \geq M + n|z| - \frac{\gamma}{2}|z|^2 \text{ and } g_t(y, z, u^2) \leq M + n|z| - \frac{\gamma}{2}|z|^2,$$

then

$$\tilde{g}_t^n(y, z, u^1) - \tilde{g}_t^n(y, z, u^2) \leq M + n|z| - \frac{\gamma}{2}|z|^2 - g_t(y, z, u^2) \leq g_t(y, z, u^1) - g_t(y, z, u^2),$$

and the desired result follows once more with the same process γ as the one for g .

Next, we show that $\tilde{g}^{n,p}$ inherits Assumption 2.5 from \tilde{g}^n . Indeed, we have

$$\tilde{g}_t^{n,p}(y, z, u^1) - \tilde{g}_t^{n,p}(y, z, u^2) \leq \sup_{(w,v) \in \mathbb{Q}^{d+1}} \{ \tilde{g}_t^n(w, v, u^1) - \tilde{g}_t^n(w, v, u^2) \},$$

which implies the desired result since the process γ in Assumption 2.5 does not depend on (y, z) .

Let now Y be a g -submartingale. We will now show that it is also a g^n -submartingale for all $n \geq 0$. Let now \mathcal{Y} (resp. \mathcal{Y}^n) be the unique solution of the BSDE with terminal condition Y_T and generator g (resp. g^n). Since g^n satisfies Assumption 2.5 and is uniformly Lipschitz in (y, z, u) , we can apply the comparison theorem for Lipschitz BSDEJs (see [34]) to obtain

$$Y_t \leq \mathcal{Y}_t \leq \mathcal{Y}_t^n, \quad \mathbb{P} - a.s.$$

Hence Y is a g^n -submartingale. We can therefore apply the Doob-Meyer decomposition in the Lipschitz case (see Theorem 1.1 in Lin [25] or Theorem 4.1 in Royer [34]) to obtain the existence of $(Z^n, U^n) \in \mathbb{H}^2 \times \mathbb{J}^2$ and of a predictable non-decreasing process A^n null at 0 such that

$$Y_t = Y_T + \int_t^T g_t^n(Y_s, Z_s^n, U_s^n) ds - \int_t^T Z_s^n dB_s - \int_t^T \int_E U_s^n(x) \tilde{\mu}(dx, ds) - A_T^n + A_t^n. \quad (3.2)$$

Since Y does not depend on n , the martingale part of (3.2) neither, which entails that Z^n and U^n are independent of n . We can rewrite (3.2) as

$$Y_t = Y_T + \int_t^T g_t^n(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds) - A_T^n + A_t^n. \quad (3.3)$$

Since g^n converges pointwise to g , the dominated convergence Theorem implies that

$$\int_0^T (g_s^n(Y_s, Z_s, U_s) - g_s(Y_s, Z_s, U_s)) ds \rightarrow 0, \mathbb{P} - a.s.$$

Hence, it holds $\mathbb{P} - a.s.$ that for all $s \in [0, T]$

$$A_s^n \rightarrow A_s := Y_s - Y_0 + \int_0^s g_r(Y_r, Z_r, U_r) dr - \int_0^s Z_r dB_r - \int_0^s \int_E U_r(x) \tilde{\mu}(dx, dr).$$

Furthermore, it is easy to see that A is still a predictable non-decreasing process null at 0.

Step 2: The concave case.

We have seen in the above proof that the main ingredients to obtain the desired decomposition are the comparison theorem and the non-linear Doob-Meyer decomposition in the Lipschitz case. As we have already seen in our comparison result of Proposition 2.7, Assumption 2.5 plays, at least formally, the same role as the concavity/convexity assumption 2.6. Moreover, we show in the Appendix (see Proposition A.1) that the non linear Doob-Meyer decomposition also holds in the Lipschitz case under Assumption 2.6 instead of Assumption 2.5. We are therefore led to proceed exactly as in the previous step. Define thus

$$\tilde{g}_t^n(y, z, u) := g_t(y, z, u) \wedge \left(M + n|z| + n\|u\|_{L^2(\nu_t)} - \frac{\gamma}{2}|z|^2 - \frac{1}{\gamma}j_t(\gamma u) \right).$$

Then \tilde{g}^n is still concave as the minimum of two concave functions, converges pointwise to g and verifies

$$-M - \beta|y| - \frac{\gamma}{2}|z|^2 - \frac{1}{\gamma}j_t(-\gamma u) \leq \tilde{g}_t^n(y, z, u) \leq M + \beta|y| + n|z| + n\|u\|_{L^2(\nu_t)}.$$

Thanks to this estimate the following sup-convolution is well defined for $p \geq \beta \vee n$

$$\tilde{g}_t^{n,p}(y, z, u) := \sup_{(w,v,r) \in \mathbb{Q}^{d+1} \times L^2(\nu_t)} \left\{ \tilde{g}_t^n(w, v, r) - p|y - w| - p|z - v| - p\|u - r\|_{L^2(\nu_t)} \right\},$$

and is still concave as the sup-convolution of concave functions.

We can then finish the proof exactly as in Step 1, using the comparison theorem of Proposition 2.7 and the non-linear Doob-Meyer decomposition given by Proposition A.1. \square

Remark 3.3. *Following the obtention of this non-linear Doob-Meyer decomposition, it is interesting to wonder whether we can say anything about the non-decreasing process A (apart from saying that it is predictable). For instance, since we are working with bounded g -supermartingales, we may think that K can also be bounded. However, it is already known for classical supermartingales (corresponding to the case $g = 0$) that this is not true. Indeed, let X be a supermartingale and let A be the predictable non-decreasing process appearing in its Doob-Meyer decomposition. Then, the inequality $|X_t| \leq M$ for all t only implies that*

$$\mathbb{E}^\mathbb{P}[(A_t)^p] \leq p!M^p, \text{ for all } p \geq 1.$$

Since we have

$$\mathbb{E}_t^\mathbb{P}[X_t - X_T] = \mathbb{E}_t^\mathbb{P}[A_T - A_t],$$

we may then wonder if there could exists another non-decreasing process C_t bounded but not necessarily adapted such that

$$\mathbb{E}_t^\mathbb{P}[X_t - X_T] = \mathbb{E}_t^\mathbb{P}[C_T - C_t]. \quad (3.4)$$

This result is then indeed true, and as shown by Meyer [27], if X is càdlàg, positive, bounded by some constant M , then if we denote \dot{X} the predictable projection of X , the non-decreasing process C in (3.4) is given by

$$C_T - C_t = M \left(1 - \exp \left(- \int_t^T \frac{dA_s^c}{M - \dot{X}_s} \right) \prod_{t < s \leq T} \left(1 - \frac{\Delta A_s}{M - \dot{X}_s} \right) \right), \quad (3.5)$$

where A^c is the continuous part of A .

If we now consider a g -supermartingale Y satisfying either one of the assumptions in Theorem 3.1, then a simple application of Itô's formula shows that

$$\tilde{Y}_t := \exp \left(\gamma Y_t + \gamma M t + \gamma \beta \int_0^t |Y_s| ds \right),$$

is a bounded classical supermartingale, which therefore admits the following decomposition

$$\tilde{Y}_t = \tilde{Y}_0 + \int_0^t \tilde{Z}_s dB_s + \int_0^t \int_E \tilde{U}_s(x) \tilde{\mu}(dx, ds) + \tilde{A}_t, \quad \mathbb{P} - a.s., \quad (3.6)$$

for some $(\tilde{Z}, \tilde{U}) \in \mathbb{H}^2 \times \mathbb{J}^2$ and some predictable non-decreasing process \tilde{A} . We can then apply Meyer's result to obtain

$$\mathbb{E}_t^{\mathbb{P}} [\tilde{Y}_t - \tilde{Y}_T] = \mathbb{E}_t^{\mathbb{P}} [D_T - D_t],$$

where D is given by (3.5).

Then, applying Itô's formula to $\ln(\tilde{Y}_t)$ in (3.6), we can show after some calculations that

$$Y_t = Y_T + \int_t^T g_t(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds) + A_T - A_t,$$

where $(Z, U) \in \mathbb{H}^2 \times \mathbb{J}^2$ and A is a predictable process with finite variation, and where (Z, U, A) can be computed explicitly from $(\tilde{Z}, \tilde{U}, \tilde{A})$.

By uniqueness of the non-linear Doob-Meyer decomposition for Y , A is actually non-decreasing, and we have a result somehow similar to that of Meyer, using the relation between \tilde{A} and A . It would of course be interesting to pursue further this study.

We end this section with a converse comparison result for our class of quadratic BSDEs, which is a consequence of the previous Doob-Meyer decomposition.

Corollary 3.1. *Let g^1 be a function satisfying either one of the assumptions in Theorem 3.1 and g^2 be another function. We furthermore suppose that $t \mapsto g_t^i(\cdot, \cdot, \cdot)$ is right continuous in $t \in [0, T]$ and continuous at T , for $i = 1, 2$. For any $\xi \in L^\infty$, denote for $i = 1, 2$, $Y_t^{i, \xi}$ the solution of the BSDEJ with generator g^i and terminal condition ξ (existence and uniqueness are assumed to hold in our spaces). If we have*

$$Y_t^{1, \xi} \leq Y_t^{2, \xi}, \quad t \in [0, T], \quad \forall \xi \in L^\infty, \quad \mathbb{P} - a.s.,$$

then we have

$$g_t^1(y, z, u) \leq g_t^2(y, z, u), \quad \forall (t, y, z, u), \quad \mathbb{P} - a.s.$$

Proof. For any $\xi \in L^\infty$, the assumption of the Corollary is equivalent to saying that $Y^{2,\xi}$ is a g^1 -supermartingale. Given the assumptions on g^1 , we can apply Theorem 3.1 to obtain the existence of $(\tilde{Z}^{2,\xi}, \tilde{U}^{2,\xi}, A^{2,\xi})$ such that for any $0 \leq s < t \leq T$

$$Y_s^{2,\xi} = Y_t^{2,\xi} + \int_s^t g_r^1 \left(Y_r^{2,\xi}, \tilde{Z}_r^{2,\xi}, \tilde{U}_r^{2,\xi} \right) dr - \int_s^t \tilde{Z}_r^{2,\xi} dB_r - \int_s^t \int_E \tilde{U}_r^{2,\xi}(x) \tilde{\mu}(dx, dr) + A_t^{2,\xi} - A_s^{2,\xi}, \quad \mathbb{P} - a.s. \quad (3.7)$$

Moreover, if we denote $(Y^{2,\xi}, Z^{2,\xi}, U^{2,\xi})$ the solution of the BSDEJ with generator g^2 and terminal condition ξ , we also have by definition

$$Y_s^{2,\xi} = Y_t^{2,\xi} + \int_s^t g_r^2 \left(Y_r^{2,\xi}, Z_r^{2,\xi}, U_r^{2,\xi} \right) dr - \int_s^t Z_r^{2,\xi} dB_r - \int_s^t \int_E U_r^{2,\xi}(x) \tilde{\mu}(dx, dr), \quad \mathbb{P} - a.s. \quad (3.8)$$

Identifying the martingale parts in (3.7) and (3.8), we obtain that $\mathbb{P} - a.s.$, $\tilde{Z}^{2,\xi} = Z^{2,\xi}$ and $\tilde{U}^{2,\xi} = U^{2,\xi}$. Furthermore, this implies by taking the expectation that

$$\frac{1}{t-s} \int_s^t \mathbb{E}^\mathbb{P} \left[g_r^1 \left(Y_r^{2,\xi}, Z_r^{2,\xi}, U_r^{2,\xi} \right) \right] dr \leq \frac{1}{t-s} \int_s^t \mathbb{E}^\mathbb{P} \left[g_r^2 \left(Y_r^{2,\xi}, Z_r^{2,\xi}, U_r^{2,\xi} \right) \right] dr.$$

Now, we finish using the same argument as in Chen [9]. Let $\xi = X_T$ where for a given (s, y_0, z_0, u_0) , X is the solution of the SDE (existence and uniqueness are classical, see for instance Jacod [14])

$$X_t = y_0 - \int_s^t g_r^2(X_r, z_0, u_0) dr + \int_s^t z_0 dB_r + \int_s^t \int_E u_0(x) \tilde{\mu}(dx, dr).$$

Letting $t \rightarrow s^+$, we obtain $g_s^1(y_0, z_0, u_0) \leq g_s^2(y_0, z_0, u_0)$, which is the desired result. \square

3.2 Upcrossing inequality

In this subsection, we prove the so-called upcrossing inequality for quadratic g -submartingales, which is similar to the one obtained by Ma and Yao [26] in the case without jumps. This property is essential for the study of path regularity of g -submartingales.

Theorem 3.2. *Let (X_t) be a g -submartingale (reps. g -supermartingale) and assume that either one of the following holds (as usual we assume existence and uniqueness for the solutions of BSDEJ driven by g with any bounded terminal condition)*

- (i) *Assumptions 2.1, 2.3(i),(ii) and 2.5 hold, with the addition that $|g(0, 0, 0)| + \alpha \leq M$, where α is the process appearing in Assumption 2.1(iii) and $M > 0$ is constant.*
- (ii) *Assumptions 2.1, 2.3(i) hold, g is concave (reap. convex), with the addition that $|g(0, 0, 0)| + \alpha \leq M$, where α is the process appearing in Assumption 2.1(iii) and $M > 0$ is constant.*

Set

$$J := \gamma M \frac{e^{\beta T} - 1}{\beta} + \gamma e^{\beta T} \|X\|_{\mathbb{D}^\infty}.$$

Denote for any $\theta \in (0, 1)$

$$\tilde{X}_t := X_t + k(J+1)t, \quad \hat{X}_t := \exp \left(k_\theta(1+J)t + \frac{\gamma\theta}{1-\theta} X_t \right) \quad t \in [0, T],$$

where k and k_θ are a well-chosen constants depending on θ , C , M , β and γ , the constants in Assumption 2.1. Let $0 = t_0 < t_1 < \dots < t_n = T$ be a subdivision of $[0, T]$ and let $a < b$, we denote $U_a^b[\tilde{X}, n]$, the number of upcrossings of the interval $[a, b]$ by $(\tilde{X}_{t_j})_{0 \leq j \leq n}$. Then

- If (i) above holds, there exists a BMO process $(\lambda_t^n, t \in [0, T])$ such that

$$\mathbb{E}^{\mathbb{P}} \left[U_a^b[X, n] \mathcal{E} \left(\int_0^T (\lambda_s^n + \phi_s) dB_s + \int_0^T \int_E \gamma_s(x) \tilde{\mu}(dx, ds) \right) \right] \leq \frac{\|X\|_{\mathcal{S}^\infty} + 2k(J+1)T + |a|}{b-a},$$

where ϕ and γ are defined in Assumption 2.3(ii) and 2.5, and such that

$$\mathbb{E}^{\mathbb{P}} \left[\int_0^{t_n} |\lambda_s^n|^2 ds \right] \leq C_1,$$

a constant independent of the choice of the subdivision.

- If (ii) above holds, then for any $\theta \in (0, 1)$

$$\mathbb{E}^{\mathbb{P}} \left[U_a^b[X, n] \right] \leq \frac{\exp \left(\frac{\gamma\theta}{1-\theta} \|X\|_{\mathcal{S}^\infty} \right) + \exp \left(\frac{\gamma\theta}{1-\theta} |a| \right)}{\exp \left(\frac{\gamma\theta}{1-\theta} b \right) - \exp \left(\frac{\gamma\theta}{1-\theta} a \right)}.$$

Proof. As usual, we can restrict ourselves to the g -submartingale case.

Step 1: When (i) holds.

For any $j \in 1, \dots, n$, we consider the following BSDE with jumps

$$Y_t^j = X_{t_j} + \int_t^{t_j} g_s(Y_s^j, Z_s^j, U_s^j) ds - \int_t^{t_j} Z_s^j dB_s - \int_t^{t_j} \int_E U_s^j(x) \tilde{\mu}(dx, ds), \quad 0 \leq t \leq t_j, \quad \mathbb{P} - a.s. \quad (3.9)$$

From Proposition 2.6 one has

$$\|Y^j\|_{\mathcal{S}^\infty} \leq \gamma M \frac{e^{\beta(t_j - t_{j-1})} - 1}{\beta} + \gamma e^{\beta(t_j - t_{j-1})} \|X_{t_j}\|_{\mathcal{S}^\infty} \leq J. \quad (3.10)$$

We can rewrite (3.9) as follows

$$\begin{aligned} Y_t^j &= X_{t_j} + \int_t^{t_j} [g_s(Y_s^j, Z_s^j, U_s^j) - g_s(Y_s^j, 0, U_s^j)] ds \\ &\quad + \int_t^{t_j} [g_s(Y_s^j, 0, U_s^j) - g_s(0, 0, U_s^j)] ds + \int_t^{t_j} [g_s(0, 0, U_s^j) - g_s(0, 0, 0)] ds \\ &\quad + \int_t^{t_j} g_s(0, 0, 0) ds - \int_t^{t_j} Z_s^j dB_s - \int_t^{t_j} \int_E U_s^j(x) \tilde{\mu}(dx, ds), \quad 0 \leq t \leq t_j, \quad \mathbb{P} - a.s. \end{aligned}$$

Then by Assumption 2.3(ii), there exist a bounded process η^n and BMO processes ϕ, λ^n with

$$|\lambda_t^n| \leq \mu \left| Z_t^j \right|, \quad \mathbb{P} - a.s., \quad \forall t \in [t_{j-1}, t_j],$$

such that

$$\begin{aligned} Y_t^j &= X_{t_j} + \int_t^{t_j} [(\lambda_s^n + \phi_s) Z_s^j + \eta_s^n Y_s^j] ds + \int_t^{t_j} [g_s(0, 0, U_s^j) - g_s(0, 0, 0)] ds \\ &\quad + \int_t^{t_j} g_s(0, 0, 0) ds - \int_t^{t_j} Z_s^j dB_s - \int_t^{t_j} \int_E U_s^j(x) \tilde{\mu}(dx, ds) \\ &= X_{t_j} + \int_t^{t_j} [\eta_s^n Y_s^j + g_s(0, 0, 0)] ds + \int_t^{t_j} [g_s(0, 0, U_s^j) - g_s(0, 0, 0)] ds \\ &\quad - \int_t^{t_j} \int_E \gamma_s(x) U_s^j(x) \nu_s(dx) ds - \int_t^{t_j} Z_s^j (dB_s - (\lambda_s^n + \phi_s) ds) \\ &\quad - \int_t^{t_j} \int_E U_s^j(x) [\tilde{\mu}(dx, ds) - \gamma_s(x) \nu_s(dx) ds] \\ &\leq X_{t_j} + k(J+1)(t_j - t) - \int_t^{t_j} Z_s^j dB_s^n - \int_t^{t_j} \int_E U_s^j(x) \tilde{\mu}_1(ds, dx), \end{aligned}$$

for some positive constant k and where

$$B_t^n := B_t - \int_0^t (\lambda_s^n + \phi_s) ds \text{ and } \tilde{\mu}_1(ds, dx) = \tilde{\mu}(dx, ds) - \gamma_s(x) \nu_s(dx) ds.$$

With our Assumptions, we can once more use Girsanov's theorem and define an equivalent probability measure \mathbb{P}^n such that

$$\frac{d\mathbb{P}^n}{d\mathbb{P}} = \mathcal{E} \left(\int_0^\cdot (\lambda_s^n + \phi_s) dB_s + \int_0^\cdot \int_E \gamma_s(x) \tilde{\mu}(dx, ds) \right)_{t_n}.$$

Taking the conditional expectation on both sides of the above inequality, we obtain

$$\mathcal{E}_t^g [X_{t_j}] = Y_t^j \leq \mathbb{E}_t^{\mathbb{P}^n} [X_{t_j}] + k(J+1)(t_j - t), \quad \mathbb{P} - a.s., \forall t \in [t_{j-1}, t_j].$$

In particular, taking $t = t_{j-1}$ we have

$$X_{t_{j-1}} \leq \mathcal{E}_{t_{j-1}}^g [X_{t_j}] \leq \mathbb{E}_{t_{j-1}}^{\mathbb{P}^n} [X_{t_j}] + k(J+1)(t_j - t_{j-1}), \quad \mathbb{P} - a.s.$$

Hence $(\tilde{X}_{t_j})_{j=0..n}$ is a \mathbb{P}^n -submartingale. Define now the quantities

$$u_t := b + k(J+1)t \text{ and } l_t := a + k(J+1)t.$$

Then, we can apply the classical upcrossing inequality for \tilde{X} , u and l

$$\mathbb{E}^{\mathbb{P}^n} [U_l^u[\tilde{X}, n]] \leq \frac{\mathbb{E}^{\mathbb{P}^n} \left[\left(\tilde{X}_T - l_T \right)^+ \right]}{u_T - l_T} \leq \frac{\|X\|_{\mathcal{S}^\infty} + 2k(J+1)T + |a|}{b - a}.$$

Notice then finally that $U_l^u[\tilde{X}, n] = U_a^b[X, n]$, which implies the desired result.

Step 2: When (ii) holds.

Using the same arguments as in the proof of (ii) of Proposition 2.7, we can show, using the concavity of g and Assumption 2.1, that for any $\theta \in (0, 1)$

$$\begin{aligned} \theta g_t(y, z, u) &\leq g_t(0, 0, 0) + C\theta |y| + (1 - \theta)M + \frac{\gamma}{1 - \theta} \theta^2 |z|^2 + \frac{1 - \theta}{\gamma} j_t \left(\frac{\gamma \theta u}{1 - \theta} \right) \\ &\leq C\theta |y| + (2 - \theta)M + \frac{\gamma}{1 - \theta} \theta^2 |z|^2 + \frac{1 - \theta}{\gamma} j_t \left(\frac{\gamma \theta u}{1 - \theta} \right). \end{aligned}$$

Hence, considering as in Step 1 for any $j = 0..n$ the solution Y^j of (3.9), we can use the same exponential transformation as in Step 2 of the proof of Proposition 2.7 to obtain

$$\begin{aligned} \exp \left(\frac{\gamma \theta}{1 - \theta} Y_t^j \right) &\leq \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(\frac{\gamma \theta}{1 - \theta} X_{t_j} + \gamma \frac{2 - \theta}{1 - \theta} M(t_j - t) + \frac{\gamma C}{1 - \theta} \int_t^{t_j} |Y_s^j| ds \right) \right] \\ &\leq \exp \left(\gamma \left(\frac{CJ}{1 - \theta} + M(2 - \theta) \right) (t_j - t) \right) \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(\frac{\gamma \theta}{1 - \theta} X_{t_j} \right) \right] \\ &\leq \exp (k_\theta (1 + J)(t_j - t)) \mathbb{E}_t^{\mathbb{P}} \left[\exp \left(\frac{\gamma \theta}{1 - \theta} X_{t_j} \right) \right], \end{aligned}$$

for some constant k_θ depending on γ , C , M and θ .

As in Step 1, choosing $t = t_{j-1}$ and using the fact that X is a g -submartingale, we deduce that $(\widehat{X}_{t_j})_{j=0..n}$ is a \mathbb{P} -submartingale, where

$$\widehat{X}_t := \exp \left(k_\theta(1+J)t + \frac{\gamma\theta}{1-\theta} X_t \right).$$

Define now the quantities

$$u_t^\theta := \exp \left(k_\theta(1+J)t + \frac{\gamma\theta}{1-\theta} b \right) \text{ and } l_t^\theta := \exp \left(k_\theta(1+J)t + \frac{\gamma\theta}{1-\theta} a \right).$$

We apply the classical upcrossing inequality for \widehat{X} , u^θ and l^θ

$$\mathbb{E}^\mathbb{P}[U_{l^\theta}^{u^\theta}[\widehat{X}, n]] \leq \frac{\mathbb{E}^\mathbb{P} \left[\left(\widehat{X}_T - l_T^\theta \right)^+ \right]}{u_T^\theta - l_T^\theta} \leq \frac{\exp \left(\frac{\gamma\theta}{1-\theta} \|X\|_{S^\infty} \right) + \exp \left(\frac{\gamma\theta}{1-\theta} |a| \right)}{\exp \left(\frac{\gamma\theta}{1-\theta} b \right) - \exp \left(\frac{\gamma\theta}{1-\theta} a \right)},$$

which ends the proof, noticing that $U_{l^\theta}^{u^\theta}[\widehat{X}, n] = U_a^b[X, n]$. \square

With this upcrossing inequality in hand, we can argue exactly as in [26] (see Corollary 5.6) to obtain

Corollary 3.2. *Let g be as in Theorem 3.2. Then any g -sub(super)martingale X admits a càdlàg modification and furthermore for any countable dense subset \mathcal{D} of $[0, T]$, it holds for all $t \in [0, T]$ that the two following limits exist \mathbb{P} -a.s.*

$$\lim_{r \uparrow t, r \in \mathcal{D}} X_r \text{ and } \lim_{r \downarrow t, r \in \mathcal{D}} X_r.$$

4 Dual Representation and Inf-Convolution

We generalize in this section some results of Barrieu and El Karoui [3] to the case of quadratic BSDEs with jumps. We give a dual representation of the related g -expectations, viewed as convex dynamic risk measures and then we compute in an explicit manner the inf-convolution of two convex g -expectations.

4.1 Dual Representation of the g -expectation

We will assume in this section that $g_t(y, z, u) = g_t(z, u)$ is independent of y and that the function g is convex. We will prove a dual Legendre-Fenchel type representation for the functional \mathcal{E}^g , making use of the Legendre-Fenchel transform of g . This problem has been treated by Barrieu and El Karoui [3] in the case of quadratic BSDEs, we extend it here to the case of quadratic BSDEs with jumps.

In this section, \mathcal{E}^g will correspond to a time consistent dynamic convex risk measures. Hence \mathcal{E}^g admits a dual representation, as in [3]. In this particular case of risk measures constructed from backward SDEs, the penalty function appearing in the dual representation is an integral of the Legendre-Fenchel transform of the generator g . The operator \mathcal{E}^g , viewed as a time-consistent dynamic convex risk measure has interesting economic applications in insurance.

For $\mu \in \mathbb{R}^d$ and $v \in L^2(\nu)$, define the Legendre-Fenchel transform of g in (z, u) as follows

$$G_t(\mu, v) := \sup_{(z, u) \in \mathbb{R}^d \times L^2(\nu_t)} \left\{ \langle \mu, z \rangle_{\mathbb{R}^d} + \langle v, u \rangle_{L^2(\nu_t)} - g_t(z, u) \right\}.$$

Let \mathcal{A} denote the space of applications $v \in \mathbb{J}_{\text{BMO}}^2 \cap L^\infty(\nu)$ such that there exists a constant $\delta > 0$ with $v_t(x) \geq -1 + \delta$, $\mathbb{P} \times dt \times d\nu_t$ -a.e.

Theorem 4.1. *Let g be a given convex function in (z, u) and let Assumptions 2.1 and 2.4 hold ; assume that $g(0, 0, 0)$ and the process α appearing in Assumption 2.1(iii) are bounded by some constant $M > 0$ (then, the existence and uniqueness of the solution of the BSDEJ with generator g and terminal condition $\xi \in L^\infty$ hold by Theorem 2.3 (ii)). We then have*

(i) *For any $\xi_T \in L^\infty$,*

$$\mathcal{E}_t^g(\xi_T) = \operatorname{ess\,sup}_{(\mu, v) \in \mathbb{H}_{\text{BMO}}^2 \times \mathcal{A}}^{\mathbb{P}} \left\{ \mathbb{E}_t^{\mathbb{Q}^{\mu, v}} \left[\xi_T - \int_t^T G_s(\mu_s, v_s) ds \right] \right\}, \quad \mathbb{P} - a.s.,$$

where $\mathbb{Q}^{\mu, v}$ is the probability measure defined by

$$\frac{d\mathbb{Q}^{\mu, v}}{d\mathbb{P}} = \mathcal{E} \left(\int_0^\cdot \mu_s dB_s + \int_0^\cdot \int_E v_s(x) \tilde{\mu}(ds, dx) \right).$$

(ii) *Moreover, there exist measurable functions $\bar{\mu}(w, t)$ and $\bar{v}(\omega, t, \cdot)$ such that*

$$\mathcal{E}_t^g(\xi_T) = \mathbb{E}_t^{\mathbb{Q}^{\bar{\mu}, \bar{v}}} \left[\xi_T - \int_t^T G_s(\bar{\mu}_s, \bar{v}_s) ds \right], \quad \mathbb{P} - a.s. \quad (4.1)$$

Proof. Thanks to the Kazamaki criterion (see for instance Lemma 4.1 in [28]), we know that if $\mu \in \mathbb{H}_{\text{BMO}}^2$ and $v \in \mathbb{J}_{\text{BMO}}^2$, then $\Gamma_{\mu, v} := \frac{d\mathbb{Q}^{\mu, v}}{d\mathbb{P}}$ is a true martingale and the probability measure $\mathbb{Q}^{\mu, v}$ is well defined.

$\mathcal{E}_t^g(\xi_T)$ is by definition solution of

$$\begin{aligned} \mathcal{E}_t^g(\xi_T) &= \xi_T + \int_t^T g_s(Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(ds, dx) \\ &= \xi_T + \int_t^T [g_s(Z_s, U_s) - \langle \mu_s, Z_s \rangle_{\mathbb{R}^d} - \langle v_s, U_s \rangle_{L^2(\nu_s)}] ds \\ &\quad - \int_t^T Z_s dB_s^\mu - \int_t^T \int_E U_s(x) \tilde{\mu}^v(ds, dx), \quad \mathbb{P} - a.s., \end{aligned} \quad (4.2)$$

where $B_t^\mu := B_t - \int_0^t \mu_s ds$ is a $\mathbb{Q}^{\mu, v}$ -Brownian motion and

$$\tilde{\mu}^v([0, t], A) := \tilde{\mu}([0, t], A) - \int_0^t \int_A v_s(x) \nu_s(dx) ds \text{ is a } \mathbb{Q}^{\mu, v} - \text{martingale.}$$

By Lemma 2.1, $Z \in \mathbb{H}_{\text{BMO}}^2$ and $U \in \mathbb{J}_{\text{BMO}}^2$. Let us prove that we also have $(Z, J) \in \mathbb{H}^2(\mathbb{Q}^{\mu, v}) \times \mathbb{J}^2(\mathbb{Q}^{\mu, v})$. Indeed, using the number $r > 1$ given by Proposition 2.4

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^{\mu, v}} \left[\int_0^T |Z_s|^2 ds \right] &= \mathbb{E}^{\mathbb{P}} \left[\Gamma_{\mu, v} \int_0^T |Z_s|^2 ds \right] \\ &\leq \left(\mathbb{E}^{\mathbb{P}} [\Gamma_{\mu, v}^r] \right)^{1/r} \left(\mathbb{E}^{\mathbb{P}} \left[\left(\int_0^T |Z_s|^2 ds \right)^q \right] \right)^{1/q} < +\infty, \end{aligned}$$

where $1/r + 1/q = 1$ and where we used Proposition 2.4 and the energy inequality (2.3). The proof for J is the same.

Moreover,

$$-G_t(\mu, v) = - \sup_{(z, u) \in \mathbb{R}^d \times L^2(\nu_t)} \left\{ \langle \mu, z \rangle_{\mathbb{R}^d} + \langle v, u \rangle_{L^2(\nu_t)} - g_t(z, u) \right\} \leq g_t(0, 0),$$

which means that $-G_t(\mu, v)$ is $\mathbb{Q}^{\mu, v} \times dt$ -integrable. Using these integrability properties and the definition of G , we take the conditional expectation in (4.2) to obtain

$$\mathcal{E}_t^g(\xi_T) \leq \mathbb{E}_t^{\mathbb{Q}^{\mu, v}} \left[\xi_T - \int_t^T G_s(\mu_s, v_s) ds \right]. \quad (4.3)$$

By our assumptions, g is C^2 in z and twice Fréchet differentiable in u , then $\partial g(Z_t, U_t)$ contains a unique element, where the subdifferential ∂g is defined by

$$\partial g(Z_t, U_t) = \left\{ (\mu, v) \in \mathbb{R}^d \times L^2(\nu_t) \text{ s.t. } g_t(z', u') \geq g_t(Z_t, U_t) - \langle \mu, z' - Z_t \rangle_{\mathbb{R}^d} - \langle v, u' - U_t \rangle_{L^2(\nu_t)}, \forall (z', u') \right\}.$$

We take $(\bar{\mu}, \bar{v}) \in \partial g(Z_t, U_t)$. We have

$$g_t(Z_t, U_t) = \langle \bar{\mu}_t, Z_t \rangle_{\mathbb{R}^d} + \langle \bar{v}_t, U_t \rangle_{L^2(\nu_t)} - G_t(\bar{\mu}_t, \bar{v}_t).$$

We refer to [3] for the measurability of $\bar{\mu}$ and \bar{v} with respect to the variable ω . We use Assumption 2.3 (which holds, since it is implied by Assumption 2.4) to write

$$\begin{aligned} |g_t(z, u)| &\leq |g_t(z, 0)| + \|D_u g_t(z, 0)\|_{L^2(\nu_t)}^2 \|u\|_{L^2(\nu_t)}^2 + C \|u\|_{L^2(\nu_t)}^2 \\ &\leq |g_t(0, 0)| + C |z|^2 + C \left(1 + \|u\|_{L^2(\nu_t)}^2 \right) \\ &\leq C |z|^2 + C \left(1 + \|u\|_{L^2(\nu_t)}^2 \right), \end{aligned}$$

where C is a constant whose value may vary from line to line. Putting the above estimation in G leads to

$$\begin{aligned} G_t(\bar{\mu}_t, \bar{v}_t) &= \sup_{(z, u) \in \mathbb{R}^d \times L^2(\nu_t)} \left\{ \langle \bar{\mu}_t, z \rangle_{\mathbb{R}^d} + \langle \bar{v}_t, u \rangle_{L^2(\nu_t)} - g_t(z, u) \right\} \\ &\geq \sup_{u \in L^2(\nu_t)} \left\{ \langle \bar{v}_t, u \rangle_{L^2(\nu_t)} - C - C \|u\|_{L^2(\nu_t)}^2 \right\} + \sup_{z \in \mathbb{R}^d} \left\{ \langle \bar{\mu}_t, z \rangle_{\mathbb{R}^d} - \frac{\gamma}{2} |z|^2 \right\} \\ &= \frac{1}{4C} \|\bar{v}_t\|_{L^2(\nu_t)}^2 - C + \frac{1}{4C} |\bar{\mu}_t|^2. \end{aligned}$$

From this, we deduce that for $\epsilon < \frac{1}{4C}$,

$$\begin{aligned} \left(\frac{1}{4C} - \epsilon \right) \left(\|\bar{v}_t\|_{L^2(\nu_t)}^2 + |\bar{\mu}_t|^2 \right) &\leq G_t(\bar{\mu}_t, \bar{v}_t) + C - \epsilon \|\bar{v}_t\|_{L^2(\nu_t)}^2 - \epsilon |\bar{\mu}_t|^2 \\ &= C - g_t(Z_t, U_t) + \langle \bar{v}_t, U_t \rangle_{L^2(\nu)} - \epsilon \|\bar{v}_t\|_{L^2(\nu)}^2 \\ &\quad + \langle \bar{\mu}_t, Z_t \rangle - \epsilon |\bar{\mu}_t|^2 \\ &\leq C - g_t(Z_t, U_t) + \frac{1}{4\epsilon} \|U_t\|_{L^2(\nu)}^2 + \frac{1}{4\epsilon} |Z_t|^2. \end{aligned}$$

Since $|g_t(Z_t, U_t)|^{\frac{1}{2}}$ and U are respectively in $\mathbb{H}_{\text{BMO}}^2$ and $\mathbb{J}_{\text{BMO}}^2$, using the fact that

$$\begin{aligned} \left(\frac{1}{4C} - \epsilon \right) \|\bar{v}_t\|_{L^2(\nu_t)}^2 &\leq \left(\frac{1}{4C} - \epsilon \right) \left(\|\bar{v}_t\|_{L^2(\nu)}^2 + |\bar{\mu}_t|^2 \right) \text{ and,} \\ \left(\frac{1}{4C} - \epsilon \right) |\bar{\mu}_t|^2 &\leq \left(\frac{1}{4C} - \epsilon \right) \left(\|\bar{v}_t\|_{L^2(\nu_t)}^2 + |\bar{\mu}_t|^2 \right), \end{aligned}$$

we obtain that \bar{v} is in $\mathbb{J}_{\text{BMO}}^2$ and $\bar{\mu}$ is in $\mathbb{H}_{\text{BMO}}^2$.

Furthermore, by our assumptions, $\bar{v} = D_u g \geq -1 + \delta$ and \bar{v} is bounded, then $\bar{v} \in \mathcal{A}$. The inequality (4.3) is thus an equality, and the representation (4.1) holds true. \square

4.2 Inf-Convolution of g -expectations

Let $g_t^1(z, u)$ and $g_t^2(z, u)$ be two convex functions such that

$$(g^1 \square g^2)(t, 0, 0) = \inf_{(\mu, v) \in \mathbb{R}^d \times L^2(\nu_t)} \{g_t^1(\mu, v) + g_t^2(-\mu, -v)\} > 0. \quad (4.4)$$

The aim of this Section is to compute the optimal risk transfer between two agents using \mathcal{E}^{g^1} and \mathcal{E}^{g^2} as risk measures. The total risk is modeled by a \mathcal{F}_T -measurable random variable ξ_T . The optimal risk transfer will be given through the inf-convolution of the risk measures \mathcal{E}^{g^1} and \mathcal{E}^{g^2} . We will show that, provided that all the quantities considered behave well enough and are in the right spaces, we can identify the inf-convolution of \mathcal{E}^{g^1} and \mathcal{E}^{g^2} as the solution of a BSDE whose generator is the inf-convolution of g^1 and g^2 . Furthermore, we will explicitly construct two \mathcal{F}_T -measurable random variables $F_T^{(1)}$ and $F_T^{(2)}$ such that $F_T^{(1)} + F_T^{(2)} = \xi_T$ and

$$(\mathcal{E}^{g^1} \square \mathcal{E}^{g^2})(\xi_T) = \mathcal{E}^{g^1}(F_T^{(1)}) + \mathcal{E}^{g^2}(F_T^{(2)}).$$

We will say that $(F_T^{(1)}, F_T^{(2)})$ is the optimal risk transfer between the agents 1 and 2.

For this purpose, and for the sake of simplicity, we will assume throughout this section that the solutions to all the considered BSDEs exists. Notice that this is not such a stringent assumption. Indeed, when it comes to the growth condition of Assumption 2.1, if we assume that g^1 has quadratic growth in z and u and is strongly convex in (z, u) , that is to say that there exists some constant $C > 0$ such that

$$g_t^1(z, u) - \frac{C}{2} \left(|z|^2 + \|u\|_{L^2(\nu_t)}^2 \right),$$

is convex, then, because g^2 is convex, it is a classical result that $g^1 \square g^2$ also has quadratic growth.

Furthermore, we are convinced that as in the classical results by Kobylanski [19] in the continuous case, this growth condition should be enough to obtain existence of maximal and minimal solutions to the corresponding BSDEs.

Remark 4.1. Notice that since our generators are defined on $\Omega \times [0, T] \times \mathbb{R}^d \times L^2(\nu) \cap L^\infty(\nu)$, a quadratic growth condition on u is equivalent to the exponential growth assumed in Assumption 2.1.

Theorem 4.2. Let g^1 and g^2 be two given generators satisfying the assumptions of Theorem 4.1. Denote $(\mathcal{E}_t^{1,2}(\xi_T), Z_t, U_t)$ the solution of the BSDE with generator $g^1 \square g^2$ and terminal condition ξ_T , and let $(\hat{Z}_t^{(1)}, \hat{U}_t^{(1)})$ and $(\hat{Z}_t^{(2)}, \hat{U}_t^{(2)})$ be four predictable processes such that

$$(g^1 \square g^2)(t, Z_t, U_t) = g_t^1(\hat{Z}_t^{(1)}, \hat{U}_t^{(1)}) + g_t^2(\hat{Z}_t^{(2)}, \hat{U}_t^{(2)}) \, dt \times \mathbb{P} - a.s. \quad (4.5)$$

Then

(i) For any \mathcal{F}_T -measurable r.v $F \in L^\infty$,

$$\mathcal{E}_t^{1,2}(\xi_T) \leq \mathcal{E}_t^{g^1}(\xi_T - F) + \mathcal{E}_t^{g^2}(F), \, \mathbb{P} - a.s., \, \forall t \in [0, T]. \quad (4.6)$$

(ii) Define

$$F_T^{(2)} := \int_0^T g_s^2(\hat{Z}_s^{(2)}, \hat{U}_s^{(2)}) ds - \int_0^T \hat{Z}_s^{(2)} dB_s - \int_0^T \int_E \hat{U}_s^{(2)}(x) \tilde{\mu}(ds, dx),$$

and assume that the BSDEs with generators g^1 and g^2 and terminal conditions $\xi_T - F_T^{(2)}$ and $F_T^{(2)}$ have a solution.

If furthermore $\hat{Z}^{(i)} \in \mathbb{H}_{\text{BMO}}^2$ and $\hat{U}^{(i)} \in \mathbb{J}_{\text{BMO}}^2$, $i = 1, 2$, then

$$(\mathcal{E}^{g^1} \square \mathcal{E}^{g^2})_t(\xi_T) = \mathcal{E}_t^{1,2}(\xi_T) = \mathcal{E}_t^{g^1}(\xi_T - F_T^{(2)}) + \mathcal{E}_t^{g^2}(F_T^{(2)}). \quad (4.7)$$

Proof. $(\hat{Z}^{(2)}, \hat{U}^{(2)})$ is well defined and predictable thanks to Proposition 8.1 in [3]. For $F \in L^\infty$, $\mathcal{E}_t^{g^1}(\xi_T - F) + \mathcal{E}_t^{g^2}(F)$ is solution of

$$\begin{aligned} d(\mathcal{E}_t^{g^1}(\xi_T - F) + \mathcal{E}_t^{g^2}(F)) &= (g_t^1(Z_t^1, U_t^1) + g_t^2(Z_t^2, U_t^2)) dt - (Z_t^1 + Z_t^2) dB_t \\ &\quad - \int_E (U_t^1(x) + U_t^2(x)) \tilde{\mu}(dt, dx) \\ &= (g_t^1(Z_t - Z_t^2, U_t - U_t^2) + g_t^2(Z_t^2, U_t^2)) dt - Z_t dB_t \\ &\quad - \int_E U_t(x) \tilde{\mu}(dt, dx) \end{aligned}$$

and $\mathcal{E}_T^{g^1}(\xi_T - F) + \mathcal{E}_T^{g^2}(F) = \xi_T$.

Since g^1 satisfies the convexity Assumption 2.6, then the function $(z, u) \mapsto g_t^1(z - Z_t^2, u - U_t^2) - g_t^2(Z_t^2, U_t^2)$ is also convex, and we can apply the comparison theorem which directly implies inequality (4.6).

Assume now that $\hat{Z}^{(i)} \in \mathbb{H}_{\text{BMO}}^2$ and $\hat{U}^{(i)} \in \mathbb{J}_{\text{BMO}}^2$, $i = 1, 2$, and define $F_t^{(i)}$ by the following forward equations

$$F_t^{(i)} := - \int_0^t g_s^i(\hat{Z}_s^{(i)}, \hat{U}_s^{(i)}) ds + \int_0^t \hat{Z}_s^{(i)} dB_s + \int_0^t \int_E \hat{U}_s^{(i)}(x) \tilde{\mu}(ds, dx).$$

Then we have

$$F_t^{(i)} = F_T^{(i)} + \int_t^T g_s^i(\hat{Z}_s^{(i)}, \hat{U}_s^{(i)}) ds - \int_t^T \hat{Z}_s^{(i)} dB_s - \int_t^T \int_E \hat{U}_s^{(i)}(x) \tilde{\mu}(ds, dx),$$

and by uniqueness, $F_t^{(i)} = \mathcal{E}_t^{g^i}(F_T^{(i)})$.

Since

$$(g^1 \square g^2)(t, Z_t, U_t) = g_t^1(\hat{Z}_t^{(1)}, \hat{U}_t^{(1)}) + g_t^2(\hat{Z}_t^{(2)}, \hat{U}_t^{(2)}) dt \times \mathbb{P} - \text{a.s.}, \quad (4.8)$$

we have the equality

$$\mathcal{E}_t^{1,2}(\xi_T) = \mathcal{E}_t^{g^1}(F_T^{(1)}) + \mathcal{E}_t^{g^2}(F_T^{(2)}).$$

Writting this last equality at time T , we obtain that $F_T^{(1)} + F_T^{(2)} = \xi_T$.

We can conclude that the processes $\mathcal{E}_t^{g^1}(F_T^{(1)}) + \mathcal{E}_t^{g^2}(F_T^{(2)})$ and $\mathcal{E}_t^{1,2}(\xi_T)$ are solution of the BSDE with coefficients $(g^1 \square g^2, \xi_T)$, by uniqueness we have that equality (4.7) holds. \square

4.3 Examples of inf-convolution

In this Section, we use the previous result on the inf-convolution of g -expectations to treat several particular examples.

4.3.1 Quadratic and Quadratic

We first study the inf-convolution of two dynamic entropic risk-measure. This example is treated by Barrieu and El Karoui [3] by a direct method, they find that the optimal risk transfer is *proportional* in the sense that there exists $a \in (0, 1)$ such that

$$(\mathcal{E}^{g^1} \square \mathcal{E}^{g^2})(\xi_T) = \mathcal{E}^{g^1}(a\xi_T) + \mathcal{E}^{g^2}((1-a)\xi_T).$$

We retrieve here this result using Theorem 4.2. For this, we first need to study the inf-convolution of the two corresponding generators g^i , $i = 1, 2$

$$g_t^i(z, u) := \frac{1}{2\gamma_i} |z|^2 + \gamma_i \int_E \left(e^{\frac{u(x)}{\gamma_i}} - 1 - \frac{u(x)}{\gamma_i} \right) \nu_t(dx), \quad (4.9)$$

where $(\gamma_1, \gamma_2) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$.

Lemma 4.1. *Let g^1 and g^2 be the two convex generators defined in equation (4.9). For any bounded \mathcal{F}_T -measurable random variable ξ_T , we have,*

$$(\mathcal{E}^{g^1} \square \mathcal{E}^{g^2})(\xi_T) = \mathcal{E}^{g^1} \left(\frac{\gamma_1}{\gamma_1 + \gamma_2} \xi_T \right) + \mathcal{E}^{g^2} \left(\frac{\gamma_2}{\gamma_1 + \gamma_2} \xi_T \right).$$

Proof.

We can calculate

$$\begin{aligned} & (g^1 \square g^2)(t, z, u) \\ &= \inf_v \left\{ \frac{1}{2\gamma_1} |v|^2 + \frac{1}{2\gamma_2} |z - v|^2 \right\} \\ &+ \inf_w \left\{ \gamma_1 \int_E \left(e^{\frac{w(x)}{\gamma_1}} - 1 - \frac{w(x)}{\gamma_1} \right) \nu_t(dx) + \gamma_2 \int_E \left(e^{\frac{u(x) - w(x)}{\gamma_2}} - 1 - \frac{u(x) - w(x)}{\gamma_2} \right) \nu_t(dx) \right\}. \end{aligned}$$

The first infimum above is easy to calculate and is attained for

$$v^* := \frac{\gamma_1}{\gamma_1 + \gamma_2} z.$$

For the second one, we postulate similarly that it should be attained for

$$w^* := \frac{\gamma_1}{\gamma_1 + \gamma_2} u.$$

In order to verify this result, it is sufficient to prove that for all $(x, y) \in \mathbb{R}^2$

$$(\gamma_1 + \gamma_2) \left(e^{\frac{x+y}{\gamma_1 + \gamma_2}} - 1 - \frac{x+y}{\gamma_1 + \gamma_2} \right) \leq \gamma_1 \left(e^{\frac{x}{\gamma_1}} - 1 - \frac{x}{\gamma_1} \right) + \gamma_2 \left(e^{\frac{y}{\gamma_2}} - 1 - \frac{y}{\gamma_2} \right). \quad (4.10)$$

Set $\lambda := \gamma_1/(\gamma_1 + \gamma_2)$, $a := x/\gamma_1$, $b := y/\gamma_2$ and $h(x) := e^x - 1 - x$, this is equivalent to

$$h(\lambda a + (1 - \lambda)b) \leq \lambda h(a) + (1 - \lambda)h(b),$$

which is clear by convexity of the function h .

Therefore, we finally obtain

$$(g^1 \square g^2)(t, z, u) = \frac{1}{2(\gamma_1 + \gamma_2)} |z|^2 + (\gamma_1 + \gamma_2) \int_E \left(e^{\frac{u(x)}{\gamma_1 + \gamma_2}} - 1 - \frac{u(x)}{\gamma_1 + \gamma_2} \right) \nu_t(dx).$$

Using the notations of Theorem 4.2, we can compute the quantity $F_T^{(2)}$, giving the optimal risk transfer

$$F_T^{(2)} = \frac{\gamma_2}{\gamma_1 + \gamma_2} \left(- \int_0^T \frac{1}{2(\gamma_1 + \gamma_2)} Z_t dt + \int_0^T Z_t dB_t + \int_0^T \int_E U_t(x) \tilde{\mu}(dt, dx) \right).$$

By definition of $F_T^{(1)}$ and $F_T^{(2)}$ and using the fact that $F_T^{(1)} + F_T^{(2)} = \xi_T$, we obtain

$$F_T^{(2)} = \frac{\gamma_2}{\gamma_1 + \gamma_2} \xi_T.$$

□

4.3.2 Linear and Quadratic

Here, we assume that $d = 1$. We study the inf-convolution of a dynamic entropic risk-measure with a linear one corresponding to a linear BSDE. In this case, we want to calculate the inf-convolution of the two corresponding generators g^1 and g^2 given by

$$g_t^1(z, u) := \frac{1}{2\gamma} |z|^2 + \gamma \int_E \left(e^{\frac{u(x)}{\gamma}} - 1 - \frac{u(x)}{\gamma} \right) \nu_t(dx),$$

and

$$g_t^2(z, u) := \alpha z + \beta \int_E (1 \wedge |x|) u(x) \nu_t(dx),$$

where $(\gamma, \alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R} \times [-1 + \delta, +\infty)$ for some $\delta > 0$.

Lemma 4.2. *Let g^1 and g^2 be defined in the two previous equations. We have, for any bounded \mathcal{F}_T -measurable random variable ξ_T ,*

$$(\mathcal{E}^{g^1} \square \mathcal{E}^{g^2})(\xi_T) = \mathcal{E}^{g^1}(F_T^{(1)}) + \mathcal{E}^{g^2}(F_T^{(2)}),$$

where

$$\begin{aligned} F_T^{(2)} = & \xi_T + \frac{1}{2} \alpha^2 \gamma T + \gamma \int_0^T \int_E (\beta(1 \wedge |x|) - \ln(1 + \beta(1 \wedge |x|))) \nu_t(dx) dt - \alpha \gamma B_T \\ & - \gamma \int_0^T \int_E \ln(1 + \beta(1 \wedge |x|)) \tilde{\mu}(dt, dx), \end{aligned}$$

and $F_T^{(1)} = \xi_T - F_T^{(2)}$.

Remark 4.2. *Notice that $F_T^{(2)}$ has no longer the linear form with respect to ξ_T obtained in the previous example. Now, the agent 2 receives the value ξ_T perturbed by a random value only depending on the data contained in the filtration, i.e the Brownian motion B and the random measures μ and (ν_t) .*

Proof. We start by computing the inf-convolution in (z, u) of the generators:

$$\begin{aligned} (g^1 \square g^2)(t, z, u) = & \inf_v \left\{ \frac{1}{2\gamma} |v|^2 + \alpha(z - v) \right\} \\ & + \inf_w \left\{ \gamma \int_E \left(e^{\frac{w(x)}{\gamma}} - 1 - \frac{w(x)}{\gamma} \right) \nu_t(dx) + \beta \int_E (1 \wedge |x|) (u(x) - w(x)) \nu_t(dx) \right\}. \end{aligned}$$

The first infimum above is easy to calculate and is attained for

$$v^* := \alpha\gamma.$$

Similarly, it is easy to show that the function $w \rightarrow \gamma \left(e^{\frac{w}{\gamma}} - 1 - \frac{w}{\gamma} \right) + \beta(1 \wedge |x|) (u(x) - w)$ attains its minimum at $w^* := \gamma \ln(1 + \beta(1 \wedge |x|))$. Therefore, we finally obtain

$$\begin{aligned} (g^1 \square g^2)(t, z, u) = & \alpha z - \frac{\alpha^2 \gamma}{2} + \int_E \beta(1 \wedge |x|) u(x) \nu_t(dx) \\ & + \int_E \gamma [\beta(1 \wedge |x|) - (1 + \beta(1 \wedge |x|)) \ln(1 + \beta(1 \wedge |x|))] \nu_t(dx). \end{aligned}$$

Notice that all the quantities appearing in $g^1 \square g^2$ are finite. Indeed, we first have for any $u \in L^2(\nu) \cap L^\infty(\nu)$

$$|\beta(1 \wedge |x|) u(x)| \leq \beta^2(1 \wedge |x|^2) + (u(x))^2, \quad (4.11)$$

and this quantity is therefore ν_t -integrable for all t . Then, since $\beta \geq -1 + \delta$, it is also clear that for some constant $C_\delta > 0$

$$0 \leq (1 + \beta(1 \wedge |x|)) \ln(1 + \beta(1 \wedge |x|)) - \beta(1 \wedge |x|) \leq C_\delta(1 \wedge |x|^2), \quad (4.12)$$

and thus the second integral is also finite.

We can now compute for instance the quantity $F_T^{(2)}$, which is given after some easy calculations by

$$\begin{aligned} F_T^{(2)} = & - \int_0^T \left(\alpha Z_t + \beta \int_E (1 \wedge |x|) U_t(x) \nu_t(dx) \right) dt + \int_0^T Z_t dB_t + \int_0^T \int_E U_t(x) \tilde{\mu}(dt, dx) \\ & + \alpha^2 \gamma T + \gamma \beta \int_0^T \int_E (1 \wedge |x|) \ln(1 + \beta(1 \wedge |x|)) \nu_t(dx) dt - \alpha \gamma B_T \\ & - \gamma \int_0^T \int_E \ln(1 + \beta(1 \wedge |x|)) \tilde{\mu}(dt, dx). \end{aligned}$$

Recall that (Z, U) is part of the solution of the BSDE with generator $g^1 \square g^2$ and terminal condition ξ_T .

Similarly, we can compute the value $F_T^{(1)}$, and using the fact that $F_T^{(1)} + F_T^{(2)} = \xi_T$, we obtain

$$\begin{aligned} \xi_T = & \frac{1}{2} \alpha^2 \gamma T + \gamma \int_0^T \int_E ((1 + \beta(1 \wedge |x|)) \ln(1 + \beta(1 \wedge |x|)) - \beta(1 \wedge |x|)) \nu_t(dx) dt \\ & - \int_0^T g_t^2(Z_t, U_t) dt + \int_0^T Z_t dB_t + \int_0^T \int_E U_t(x) \tilde{\mu}(dt, dx). \end{aligned}$$

And finally, we can conclude that the optimal risk transfer takes the form

$$\begin{aligned} F_T^{(2)} = & \xi_T + \frac{1}{2} \alpha^2 \gamma T + \gamma \int_0^T \int_E (\beta(1 \wedge |x|) - \ln(1 + \beta(1 \wedge |x|))) \nu_t(dx) dt - \alpha \gamma B_T \\ & - \gamma \int_0^T \int_E \ln(1 + \beta(1 \wedge |x|)) \tilde{\mu}(dt, dx). \end{aligned}$$

□

4.3.3 Absolute value and Quadratic

Here, we study again the example of the entropic risk measure, i.e. we keep the generator g^1 given by

$$g_t^1(z, u) := \frac{1}{2\gamma} |z|^2 + \gamma \int_E \left(e^{\frac{u(x)}{\gamma}} - 1 - \frac{u(x)}{\gamma} \right) \nu_t(dx),$$

where $\gamma > 0$. The generator g^2 correspond to the second example of risk measure given in Examples 3.1, i.e.

$$g_t^2(z, u) := \eta |z| + \eta \int_E (1 \wedge |x|) u^+(x) \nu_t(dx) - C_1 \int_E (1 \wedge |x|) u^-(x) \nu_t(dx),$$

where $\eta > 0$ and $-1 + \delta \leq C_1 \leq 0$ for some $\delta > 0$.

Then

$$\begin{aligned} (g^1 \square g^2)(t, z, u) &= \inf_v \left\{ \frac{1}{2\gamma} |v|^2 + \eta |z - v| \right\} \\ &+ \inf_w \left\{ \gamma \int_E \left(e^{\frac{w(x)}{\gamma}} - 1 - \frac{w(x)}{\gamma} \right) \nu_t(dx) + \eta \int_E (1 \wedge |x|) (u(x) - w(x)^+) \nu_t(dx) \right. \\ &\left. - C_1 \int_E (1 \wedge |x|) (u(x) - w(x)^-) \nu_t(dx) \right\}. \end{aligned}$$

These infima are easy to calculate and are attained at (v^*, w^*) defined by

$$\begin{aligned} v^* &= \mu\gamma \mathbf{1}_{\{z > \mu\gamma\}} - \mu\gamma \mathbf{1}_{\{z < -\mu\gamma\}} + z \mathbf{1}_{\{-\mu\gamma \leq z \leq \mu\gamma\}} \\ w^*(x) &= \psi(\eta, x) \mathbf{1}_{\{u(x) > \psi(\eta, x)\}} + \psi(C_1, x) \mathbf{1}_{\{u(x) \leq \psi(C_1, x)\}} + u(x) \mathbf{1}_{\{\psi(C_1, x) < u(x) \leq \psi(\eta, x)\}}. \end{aligned}$$

where $\psi(r, x) := \gamma \ln(1 + r(1 \wedge |x|))$. We finally obtain

$$(g^1 \square g^2)(t, z, u) = A(z) + B(u),$$

where

$$\begin{aligned} A(z) &:= \left(\mu z - \frac{1}{2} \mu^2 \gamma \right) \mathbf{1}_{\{z > \mu\gamma\}} - \left(\mu z + \frac{1}{2} \mu^2 \gamma \right) \mathbf{1}_{\{z < -\mu\gamma\}} + \frac{1}{2\gamma} z^2 \mathbf{1}_{\{-\mu\gamma \leq z \leq \mu\gamma\}} \\ B(u) &:= \int_E [C_1(1 \wedge |x|)(\gamma + u(x)) - \psi(C_1, x)(1 + C_1(1 \wedge |x|)) \mathbf{1}_{\{u(x) \leq \psi(C_1, x)\}}] \nu_t(dx) \\ &+ \int_E [\eta(1 \wedge |x|)(\gamma + u(x)) - \psi(\eta, x)(1 + \eta(1 \wedge |x|)) \mathbf{1}_{\{u(x) > \psi(\eta, x)\}}] \nu_t(dx) \\ &+ \int_E \left[\gamma \left(e^{\frac{u(x)}{\gamma}} - 1 - \frac{u(x)}{\gamma} \right) \mathbf{1}_{\{\psi(C_1, x) < u(x) \leq \psi(\eta, x)\}} \right] \nu_t(dx), \end{aligned}$$

that is to say this inf-convolution is equal to a linear function outside a given set, and is quadratic inside the set (in z and u). Notice that all the quantities appearing in $B(u)$ are finite, thanks to the inequalities (4.11) and (4.12) given in the previous example.

Now to have the optimal risk transfer, we calculate $F_T^{(1)}$ and $F_T^{(2)}$ using their definition. We obtain

for example for $F_T^{(2)}$

$$\begin{aligned}
F_T^{(2)} &= \int_0^T (\mu^2 \gamma - \mu Z_t) \mathbf{1}_{\{|Z_t| > \mu \gamma\}} dt \\
&\quad - \int_0^T \int_E \left[\eta(1 \wedge |x|) (\psi(\eta, x) \mathbf{1}_{\{U_t(x) > \psi(\eta, x)\}} + U_t(x) \mathbf{1}_{\{0 < U_t(x) < \psi(\eta, x)\}}) \right. \\
&\quad \left. + C_1(1 \wedge |x|) (\psi(C_1, x) \mathbf{1}_{\{U_t(x) < \psi(C_1, x)\}} + U_t(x) \mathbf{1}_{\{\psi(C_1, x) < U_t(x) < 0\}}) \right] \nu_t(dx) dt \\
&\quad + \int_0^T \int_E \left(U_t(x) - \psi(\eta, x) \mathbf{1}_{\{U_t(x) > \psi(\eta, x)\}} + \psi(C_1, x) \mathbf{1}_{\{U_t(x) \leq \psi(C_1, x)\}} \right. \\
&\quad \left. + U_t(x) \mathbf{1}_{\{\psi(C_1, x) < U_t(x) \leq \psi(\eta, x)\}} \right) \tilde{\mu}(dt, dx),
\end{aligned}$$

where (Z, U) is part of the solution of the BSDE with generator $g^1 \square g^2$ and terminal condition ξ_T .

And we have of course $F_T^{(1)} = \xi_T - F_T^{(2)}$.

Remark 4.3. In this example, the optimal transfer has again no longer the proportional form found in example 1. Moreover, this time we are only able to calculate $F_T^{(1)}$ and $F_T^{(2)}$ in function of (Z, U) , where, as defined before, $(\mathcal{E}_t^{1,2}(\xi_T), Z_t, U_t)$ denotes the solution of the BSDE with generator $g^1 \square g^2$ and terminal condition ξ_T .

A Appendix

Proposition A.1. Let g be a function satisfying Assumption 2.1(i), which is uniformly Lipschitz in (y, z, u) . Let Y be a càdlàg g -submartingale (resp. g -supermartingale) in \mathcal{S}^∞ . Assume further that g is concave (resp. convex) in (z, u) and that

$$\begin{aligned}
g_t(y, z, u) &\geq -M - \beta |y| - \frac{\gamma}{2} |z|^2 - \frac{1}{\gamma} j_t(-\gamma u). \\
\left(\text{resp. } g_t(y, z, u) &\leq M + \beta |y| + \frac{\gamma}{2} |z|^2 + \frac{1}{\gamma} j_t(\gamma u) \right).
\end{aligned}$$

Then there exists a predictable non-decreasing (resp. non-increasing) process A null at 0 and processes $(Z, U) \in \mathbb{H}^2 \times \mathbb{J}^2$ such that

$$Y_t = Y_T + \int_t^T g_s(Y_s, Z_s, U_s) ds - \int_t^T Z_s dB_s - \int_t^T \int_E U_s(x) \tilde{\mu}(dx, ds) - A_T + A_t, \quad t \in [0, T].$$

Proof. As usual, we can limit ourselves to the g -supermartingale case. Let us consider the following reflected BSDEJ with lower obstacle Y , generator g and terminal condition Y_T

$$\begin{aligned}
\tilde{Y}_t &= Y_T + \int_t^T g_s(\tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) ds - \int_t^T \tilde{Z}_s dB_s - \int_t^T \int_E \tilde{U}_s(x) \tilde{\mu}(dx, ds) + A_T - A_t, \quad t \in [0, T], \quad \mathbb{P} - a.s. \\
\tilde{Y}_t &\geq Y_t, \quad t \in [0, T], \quad \mathbb{P} - a.s. \\
\int_0^T (\tilde{Y}_{s-} - Y_{s-}) dA_s &= 0, \quad \mathbb{P} - a.s.
\end{aligned} \tag{A.1}$$

The existence and uniqueness of a solution consisting of a predictable non-decreasing process A and a triplet $(\tilde{Y}, \tilde{Z}, \tilde{U}) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{J}^2$ follow from the results of [12] for instance, since the generator is Lipschitz and the obstacle is càdlàg and bounded.

We will now show that the process \tilde{Y} must always be equal to the lower obstacle Y , which will provide us the desired decomposition. We proceed by contradiction and assume without loss of generality that $\tilde{Y}_0 > Y_0$. For any $\varepsilon > 0$, we now define the following bounded stopping time

$$\tau_\varepsilon := \inf \left\{ t > 0, \tilde{Y}_t \leq Y_t + \varepsilon, \mathbb{P} - a.s. \right\} \wedge T.$$

By the Skorokhod condition, it is a classical result that the non-decreasing process A never acts before τ_ε . Therefore, we have for any $t \in [0, \tau_\varepsilon]$

$$\tilde{Y}_t = \tilde{Y}_{\tau_\varepsilon} + \int_t^{\tau_\varepsilon} g_s(\tilde{Y}_s, \tilde{Z}_s, \tilde{U}_s) ds - \int_t^{\tau_\varepsilon} \tilde{Z}_s dB_s - \int_t^{\tau_\varepsilon} \int_E \tilde{U}_s(x) \tilde{\mu}(dx, ds), \quad \mathbb{P} - a.s. \quad (\text{A.2})$$

Consider now the BSDEJ on $[0, \tau_\varepsilon]$ with terminal condition Y_{τ_ε} and generator g (existence and uniqueness of the solution are consequences of the result of [1] or [24])

$$\hat{Y}_t = Y_{\tau_\varepsilon} + \int_t^{\tau_\varepsilon} g_s(\hat{Y}_s, \hat{Z}_s, \hat{U}_s) ds - \int_t^{\tau_\varepsilon} \hat{Z}_s dB_s - \int_t^{\tau_\varepsilon} \int_E \hat{U}_s(x) \hat{\mu}(dx, ds), \quad \mathbb{P} - a.s. \quad (\text{A.3})$$

Notice also that since Y and $g(0, 0, 0)$ are bounded, \hat{Y} and \tilde{Y} are also bounded, as a consequence of classical *a priori* estimates for Lipschitz BSDEJs and RBSDEJs. Then, using the fact that g is convex in (z, u) and that

$$g_t(y, z, u) \leq M + \beta |y| + \frac{\gamma}{2} |z|^2 + \frac{1}{\gamma} j_t(\gamma u),$$

we can proceed exactly as in the Step 2 of the proof of Proposition 2.7 to obtain that for any $\theta \in (0, 1)$

$$\begin{aligned} \tilde{Y}_0 - \theta \hat{Y}_0 &\leq \frac{1 - \theta}{\gamma} \ln \left(\mathbb{E}^\mathbb{P} \left[\exp \left(\gamma \int_0^{\tau_\varepsilon} \left(M + C |\hat{Y}_t| + \frac{C}{1 - \theta} |\tilde{Y}_t - \theta \hat{Y}_t| \right) ds + \frac{\gamma}{1 - \theta} (\tilde{Y}_{\tau_\varepsilon} - \theta Y_{\tau_\varepsilon}) \right) \right] \right) \\ &\leq (1 - \theta) \ln(C_0) + C \left(\int_0^{\tau_\varepsilon} \|\tilde{Y}_s - \theta \hat{Y}_s\|_\infty ds + \|\tilde{Y}_{\tau_\varepsilon} - \theta Y_{\tau_\varepsilon}\|_\infty \right), \end{aligned}$$

where C_0 is some constant which does not depend on θ .

Letting θ go to 1, and using the fact that by definition $\tilde{Y}_{\tau_\varepsilon} - \theta Y_{\tau_\varepsilon} \leq \varepsilon$, we obtain

$$\tilde{Y}_0 - \hat{Y}_0 \leq C\varepsilon + C \int_0^{\tau_\varepsilon} \|\tilde{Y}_s - \hat{Y}_s\|_\infty ds. \quad (\text{A.4})$$

But since Y is a g -supermartingale, we have by definition that $\hat{Y}_0 \leq Y_0$. Since we assumed that $\tilde{Y}_0 > Y_0$, this implies that $\tilde{Y}_0 > \hat{Y}_0$. Therefore, we can use Gronwall's lemma in (A.4) to obtain

$$0 < \tilde{Y}_0 - \hat{Y}_0 \leq C_1 \varepsilon,$$

for some constant $C_1 > 0$, independent of ε .

By arbitrariness of ε , this gives us the desired contradiction and ends the proof. \square

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